Notes by-

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Dimensional Analysis [DA]

Dimensional Analysis is a mathematical tool which uses study of dimensions as an aid to solutions of many Engg. problems.

Dimension :- Expression for derived quantity in terms of primary quantities is called as dimension of physical quantity.

Fouriers Principle of dimension homogenity :-

An eqⁿ which expresses a physical phenomenon of fluid flow must be algebrically correct & dimensionally homogeneous.

Mtds of DA : 1) **RayLeigh's Mtd**: Every variable has diff. dimensions hence they cannot be added of substracted.

eg: $X = f(X_1, X_2, X_3, \dots, X_n)$

X ↑ Dependent Variable; $X_1, \dots, X_n$ Independent variable.

Then $X = C \cdot X_1^{a} \cdot X_2^{b} \cdot X_3^{c} \cdots X_n^{n}$

where C = Dimensionless const. which can be determined from physical cht. of problem or experimental measurement.

Then the eqⁿ is expressed in dimensions [~~M L~~ T]

$$[M^x L^y T^z] = [M^0 L^0 T^0] \cdot [M^p L^q T^r]^a \, [M^s L^t T^u]^b \cdot [M^l L^m T^n]^c$$

For eqⁿ to be dimensionally ballanced,

$$x = p \cdot a + s \cdot b + r \cdot c$$
$$y = q \cdot a + t\, b + m \cdot c$$
$$z = r \cdot a + u \cdot b + n \cdot c$$

} Where a, b, c are unknowns which can be find by ... 3 unknown = 3 Eqⁿ.

Pro: Q = Discharge thr orifice, D = Dia. of orifice, H = const. head, ρ = Mass density, μ = Dynamic Viscosity, g = gravitational acceln.

M L T
kg m s

$$Q = f(\mu, \rho, d, H, g)$$

$$\therefore Q = C \cdot \mu^a \rho^b d^c H^d g^e \quad \text{(i)}$$

$$\therefore [M^0 L^3 T^{-1}] = [M^1 L^{-1} T^{-1}]^a [M^1 L^{-3} T^0]^b [M^0 L^1 T^0]^c [M^0 L^1 T^0]^d [M^0 L^1 T^{-2}]^e$$

m³/s

$$\therefore 0 = a + b \Rightarrow \boxed{b = -a}$$
$$3 = -a - 3b + c + d + e$$
$$-1 = -a - 2e \Rightarrow \boxed{\frac{1-a}{2} = e}$$

μ = N·s/m²

$$\therefore 3 = -a + 3a + c + d + \frac{1}{2} - \frac{a}{2} \Rightarrow \boxed{c = \frac{5}{2} - \frac{3}{2}a - d}$$

Eqⁿ (i) becomes, $Q = C \cdot \mu^a \cdot \rho^{-a} \cdot d^{(5/2 - 3/2 a - d)} \cdot H^d \cdot g^{1/2 - \frac{a}{2}}$

$$\boxed{\therefore Q = C \left[\frac{\mu}{\rho \cdot d^{3/2} g^{1/2}}\right]^a \cdot (d^{5/2} g^{1/2}) \cdot \left(\frac{H}{d}\right)^d}$$

Imp

*** **Buckingham-π Method** :-

Buckingham π Theorem: If 'n' dimensional variables are there which can be described by 'm' fundamental quantities or dimensions [MLT] & are related by dimensionaly homogeneous eqⁿ then relationship amoung 'n' quantities can always be expressed in terms of exactly (n-m) terms, called as non dimensional 'π' terms.

eg. $f'(F_D, D, V, \rho, \mu) = 0$ $n = 5$, $m = [M\ L\ T] = 3$

$\therefore n - m = 5 - 3 = 2$

Nondimensional term = π terms. $\therefore$ 2 π terms are reqd.

i.e. $\phi(\pi_1, \pi_2) = 0$ To find π terms ~~two~~ 9 variables are divided into two groups. ρ, D, V are primary variable.

$$\therefore \pi_1 = (\rho^{a_1} D^{b_1} V^{c_1})\, \mu^{d_1} \quad \text{—— ①}$$

$$\pi_2 = (\rho^{a_2} D^{b_2} V^{c_2})\, F_D^{d_2} \quad \text{—— ②}$$

$$\therefore [M^0 L^0 T^0] = [M^1 L^{-3} T^0]^{a_1} [M^0 L^1 T^0]^{b_1} [M^0 L^1 T^{-1}]^{c_1} [M^1 L^{-1} T^{-1}]^{d_1} \ldots \text{ from ①}$$

$0 = a_1 + d_1 \Rightarrow a_1 = -d_1$

$0 = -3a_1 + b_1 + c_1 - d \Rightarrow b_1 = -d_1$

$0 = -c_1 - d_1 \Rightarrow c_1 = -d_1$

$N = m \cdot a$
$= kg \cdot m/s^2$
$= [M^1 L^1 T^{-2}]$

$$\therefore \pi_1 = \rho^{-d_1} D^{-d_1} V^{-d_1} \cdot \cancel{F_D}\, \mu^{d_1}$$

$$\therefore \pi_1 = \left[\frac{\mu}{\rho \cdot V \cdot D}\right]^{d_1} = \left[\frac{1}{Re}\right]^{d_1} = [Re]^{-d_1}$$

$$[M^0 L^0 T^0] = [M^1 L^{-3} T^0]^{a_2} [M^0 L^1 T^0]^{b_2} [M^0 L^1 T^{-1}]^{c_2} [M^1 L^1 T^{-2}]^{d_2}$$

$\therefore 0 = a_2 + d_2 \Rightarrow a_2 = -d_2$

$0 = -3a_2 + b_2 + c_2 + d_2 \Rightarrow +3d_2 + b_2 - 2d_2 + d_2 = 0 \Rightarrow b_2 + 2d_2 = 0 \Rightarrow b_2 = -2d_2$

$0 = -c_2 - 2d_2 \Rightarrow c_2 = -2d_2$

$$\therefore \pi_2 = \rho^{-d_2} \cdot D^{-2d_2} \cdot V^{-2d_2} \cdot F_D^{d_2}$$

$$\therefore \pi_2 = \left[\frac{F_D}{\rho \cdot V^2 \cdot D^2}\right]^{d_2}$$

* Similarity :- 1) Geometric Similarity 2) Kinematic similarity 3) Dynamic similarity.
  - → Length Scale Ratio $= L_r = \dfrac{L_m \text{ — model}}{L_p \text{ — prototype}} = \dfrac{b_m}{b_p} = \dfrac{d_m}{d_p}$
  - → Area Scale Ratio $= A_r = \dfrac{A_m}{A_P} = \dfrac{L_m \cdot b_m}{L_p \times b_p} \times \ldots = L_r^2$
  - → Volume scale Ratio $= V_r = L_r^3$
  - → Time Scale Ratio $= T_r = T_m / T_p$
  - → Vel. Scale Ratio $= V_r = V_m / V_p = \dfrac{L_m/T_m}{L_p/T_p} = \dfrac{L_r}{T_r}$
  - → Accel^n Scale Ratio $= A_r = \dfrac{L_r}{T_r^2}$
  - → Discharge Scale Ratio $= Q_r = L_r^3 / T_r$
  - → Force Ratio $= \dfrac{(\Sigma F)_m}{(\Sigma F)_p} = \dfrac{(F_g + F_p + F_v + F_e + F_s)_m}{(F_g + F_p + F_v + F_e + F_s)_p}$

① Inertia force Ratio: $(F_i = \rho \cdot AV \cdot V = \rho \cdot L^2 \cdot V^2)$ $\therefore F_i$ Ratio $= \dfrac{(F_i)_m}{(F_i)_p} = \dfrac{\rho_m L_m^2 V_m^2}{\rho_p\, L_p^2 \cdot V_m^2} = \rho_r \cdot L_r \cdot V_r$

② Inertia to Viscous force Ratio $\Rightarrow F_i = \rho \cdot L^2 V^2$ & $F_v = \mu \cdot V \cdot L$; Ratio $= \dfrac{F_i}{F_V} = \dfrac{\rho \cdot L^2 V^2}{\mu \cdot V \cdot L} = \dfrac{\rho V L}{\mu} = Re$

③ Inertia to gravity force Ratio $\Rightarrow \dfrac{F_i}{F_g} = \dfrac{\rho \cdot L^2 V^2}{\rho \cdot L^3 \cdot g} = \dfrac{V^2}{Lg} \Rightarrow \dfrac{V}{\sqrt{gL}} = Fr.$

④ Inertia to pressure force Ratio $\Rightarrow \dfrac{F_i}{F_P} = \dfrac{\rho \cdot L^2 \cdot V^2}{p \cdot L^2} = \dfrac{V^2}{P/\rho} \Rightarrow \dfrac{V}{\sqrt{P/\rho}}$ = Eulers No $= \dfrac{1}{\text{Newtons No}}$

⑤ Inertia to elasticity force Ratio $\Rightarrow \dfrac{F_i}{F_e} = \dfrac{\rho \cdot L^2 V^2}{K \cdot L^2} = \dfrac{V^2}{K/\rho} \Rightarrow \dfrac{V}{\sqrt{K/\rho}}$ = Mach No. $= \dfrac{V^2}{C^2}$

Where $C = \sqrt{K/\rho}$ = Vel. of Sound.

$\therefore$ $Ma > 1 \Rightarrow$ Supersonic
$Ma >> 1 \Rightarrow$ Hypersonic
$Ma < 1 \Rightarrow$ Subsonic
$Ma = 1 \Rightarrow$ Sonic

⑥ Inertia to Surface tension Ratio $\Rightarrow \dfrac{F_i}{F_s} = \dfrac{\rho \cdot L^2 V^2}{\sigma \cdot L} = \dfrac{V^2}{\sigma/\rho L} = \dfrac{V}{\sqrt{\sigma/\rho L}}$ = Weber No.

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