ENGINEERING MECHANICS

Notes by-

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Applied mechanics

EM-I

MATHEMATICAL PREREQUISITES

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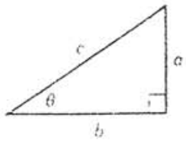
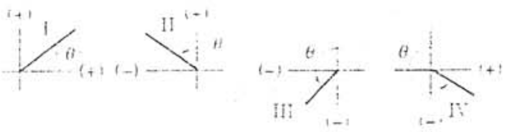
TRIGONOMETRY

1. Definitions

$\sin\theta = a/c \quad \csc\theta = c/a$
$\cos\theta = b/c \quad \sec\theta = c/b$
$\tan\theta = a/b \quad \cot\theta = b/a$



2. Signs in the four quadrants



	I	II	III	IV
$\sin\theta$	−	+	−	−
$\cos\theta$	−	−	−	+
$\tan\theta$	−	−	+	−

Note: Limit the use of *quadrants*. Have a thorough understanding of the 'trigonometric waves'. **Never** express the direction of a force, velocity, etc in terms of quadrants.

3. Miscellaneous relations

$\sin^2\theta + \cos^2\theta = 1$
$1 + \tan^2\theta = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

$\sin\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1 - \cos\theta)}$ $\qquad \cos\theta = 1 - 2\sin^2\theta/2$
$\qquad \cos\theta = 2\cos^2\theta/2 - 1$

$\cos\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1 + \cos\theta)}$

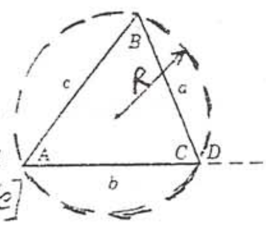
$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$
$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$

4. Law of sines

$\frac{a}{b} = \frac{\sin A}{\sin B}$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

[R is the radius of the circumcircle]

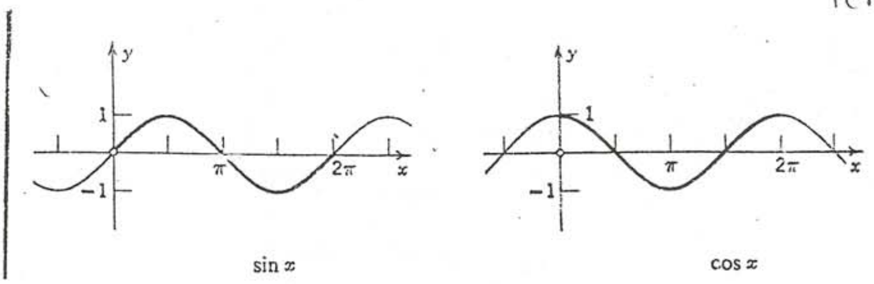


5. Law of cosines

$c^2 = a^2 + b^2 - 2ab\cos C$
$c^2 = a^2 + b^2 + 2ab\cos D$

Sine and cosine functions †In calculus, angles are measured in radians, so that $\sin x$ and $\cos x$ have period 2π. $\sin x$ is odd, $\sin(-x) = -\sin x$, and $\cos x$ is even, $\cos(-x) = \cos x$.

Period of $\sin f^n = 2\pi$
$\tan f^n = \pi$



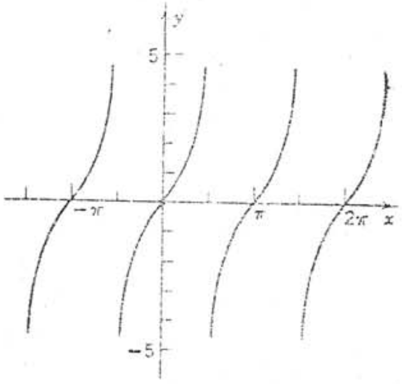
$\sin x$ $\qquad$ $\cos x$

† Note the fact that *angles in degrees(°)* and *logarithm to the base 10(log₁₀)* are, in a strict sense, not mathematically acceptable even though we use them quite often. We have 'invented' these for our intuitive convenience.

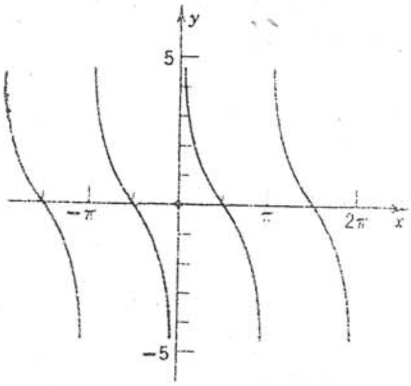
Tangent, cotangent, secant, cosecant

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \quad \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$



$\tan x$



$\cot x$

Hyperbolic functions (hyperbolic sine $\sinh x$, etc.;

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}), \qquad \cosh x = \tfrac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}$$

$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1), \qquad \cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1)$$

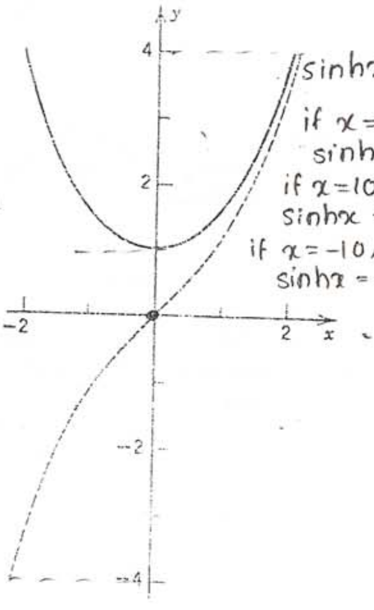
$$\begin{cases} \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \\ \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \end{cases}$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$\cosh x = \dfrac{e^{x}+e^{-x}}{2}$

$x = 0$, $\cosh x = 1$

$x = +ve$, $\cosh x = +ve$ large



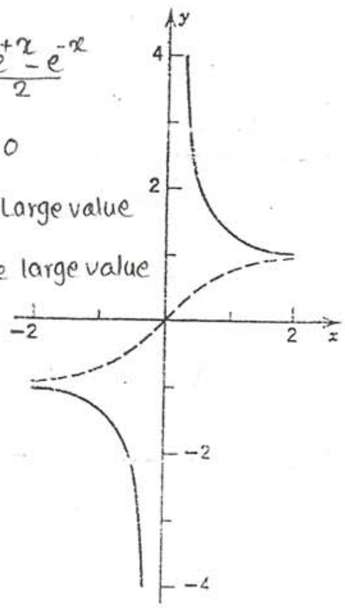
$\sinh x = \dfrac{e^{+x} - e^{-x}}{2}$

if $x = 0$,
$\sinh x = 0$

if $x = 10$,
$\sinh x$ = + Large value

if $x = -10$,
$\sinh x$ = −ve large value

$\sinh x$ (dashed) and $\cosh x$



$\tanh x$ (dashed) and $\coth x$

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INTEGRALS

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{dx}{x} = \ln x$$

$$\int \sqrt{a + bx} \, dx = \frac{2}{3b} \sqrt{(a + bx)^3}$$

$$\int e^{ax} \, dx = \frac{e^{ax}}{a}$$

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\int \sec x \, dx = \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x}$$

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4}$$

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4}$$

$$\int \sin x \cos x \, dx = \frac{\sin^2 x}{2}$$

$$\int \sinh x \, dx = \cosh x$$

$$\int \cosh x \, dx = \sinh x$$

$$\int \tanh x \, dx = \ln \cosh x$$

$$\int \ln x \, dx = x \ln x - x$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln [f(x)]$$

An important note

Numerical techniques are widely used in Engineering for the solution of differential equations, Runga-Kutta methods being the most 'popular' ones. There is, however, a dangerous tendency among many engineers to adopt these techniques, ignorant of the fact that closed-form solutions are available. A substantial variety of problems in Mechanics can be solved with a mastery over *ordinary differential equations of the first order* and *ordinary linear differential equations.* Especially useful in Dynamics are

$$\frac{dy}{dx} + f(x)\, y = \phi(x)$$

and

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$

Very often, some 'tricky' substitutions may simplify a DE to one of the above-mentioned DEs.

INTEGRALS (Continued)

$$\frac{dx^n}{dx} = nx^{n-1}, \quad \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \quad \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\lim_{\Delta x \to 0} \sin \Delta x = \sin dx = \tan dx = dx$$

$$\lim_{\Delta x \to 0} \cos \Delta x = \cos dx = 1$$

$$\frac{d \sin x}{dx} = \cos x, \quad \frac{d \cos x}{dx} = -\sin x, \quad \frac{d \tan x}{dx} = \sec^2 x$$

$$\frac{d \sinh x}{dx} = \cosh x, \quad \frac{d \cosh x}{dx} = \sinh x, \quad \frac{d \tanh x}{dx} = \operatorname{sech}^2 x$$

$$\int \sqrt{a^2 - x^2}\, dx = \tfrac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right)$$

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2\sqrt{a + bx}}{b}$$

* These Integrals are frequently encountered in *Dynamics*.

* $$\int \frac{dx}{a \pm bx^2} = \frac{1}{\sqrt{ab}} \begin{matrix}\tan^{-1} \\ \tanh^{-1}\end{matrix} \frac{x\sqrt{ab}}{a} \quad \text{or} \quad \frac{1}{\sqrt{-ab}} \tanh^{-1} \frac{x\sqrt{-ab}}{a}$$

* $$\int \frac{x\, dx}{a + bx^2} = \frac{1}{2b} \ln (a + bx^2)$$

* $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right)$$

$$\int x \sin x\, dx = \sin x - x \cos x$$

$$\int x \cos x\, dx = \cos x + x \sin x$$

* $$\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}\left[a+bx-a\ln(a+bx)\right]$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln (x + \sqrt{x^2 \pm a^2})$$

* $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$$

* $$\int e^{ax} \sin px\, dx = \frac{e^{ax}(a \sin px - p \cos px)}{a^2 + p^2}$$

$$\int \frac{x\, dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}$$

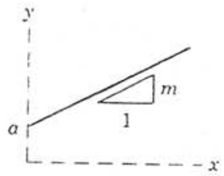
* $$\int e^{ax} \cos px\, dx = \frac{e^{ax}(a \cos px + p \sin px)}{a^2 + p^2}$$

$$\int \frac{x\, dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2}$$

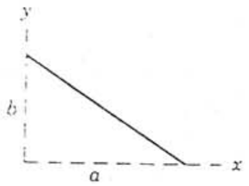
$$\int \frac{dx}{a - bx^2} = \frac{1}{\sqrt{ab}} \tanh^{-1}\left(\frac{x\sqrt{ab}}{a}\right)$$

Analytic Geometry

1. Straight line

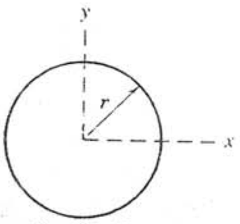


$y = a + mx$

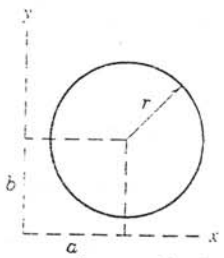


$\frac{x}{a} + \frac{y}{b} = 1$

2. Circle

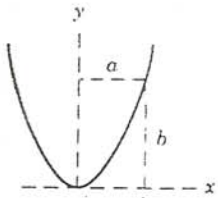


$x^2 + y^2 = r^2$

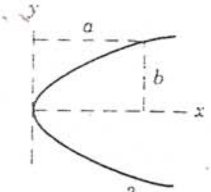


$(x-a)^2 + (y-b)^2 = r^2$

3. Parabola

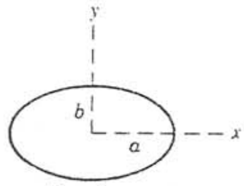


$y = b\frac{x^2}{a^2}$



$x = a\frac{y^2}{b^2}$

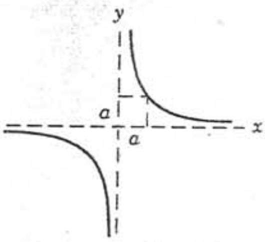
4. Ellipse



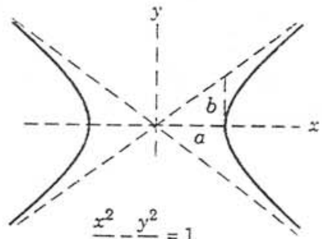
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

In case 'perimeters' of any closed curve or 'length' of a curve is required to be found, perform $\int \sqrt{1+(dy/dx)^2}\, dx$ in the required domain.

5. Hyperbola



$xy = a^2$

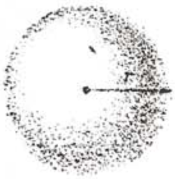


$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solid Geometry

1. Sphere

Volume $= \frac{4}{3}\pi r^3$
Surface area $= 4\pi r^2$



2. Spherical wedge

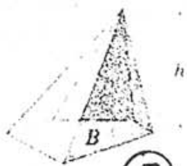
Volume $= \frac{2}{3}r^3\theta$

3. Right circular cone

Volume $= \frac{1}{3}\pi r^2 h$
Lateral area $= \pi r L$
$L = \sqrt{r^2 + h^2}$

4. Any pyramid or cone

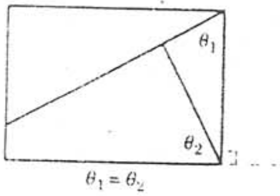
Volume $= \frac{1}{3}Bh$
where B = area of base



(5)

PLANE GEOMETRY

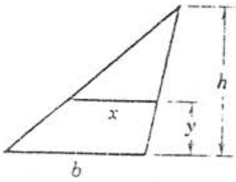
1. When two intersecting lines are, respectively, perpendicular to two other lines, the angles formed by the two pairs are equal.



$\theta_1 = \theta_2$

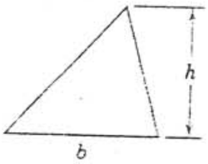
2. Similar triangles

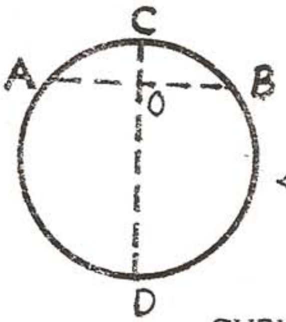
$$\frac{x}{b} = \frac{h - y}{h}$$



3. Any triangle

Area $= \frac{1}{2}bh$

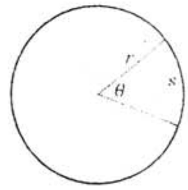




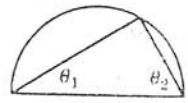
If CD is diameter
AB is ⊥r to CD
$OA \cdot OB = OC \cdot OD$

4. Circle

Circumference $= 2\pi r$
Area $= \pi r^2$
Arc length $s = r\theta$
Sector area $= \frac{1}{2}r^2\theta$



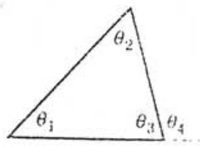
5. Every triangle inscribed within a semicircle is a right triangle.



$\theta_1 + \theta_2 = \pi/2$

6. Angles of a triangle

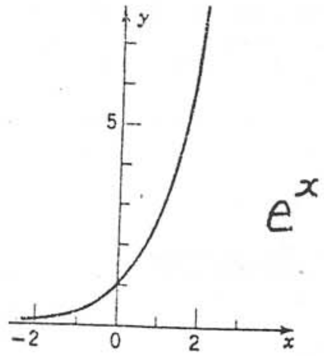
$\theta_1 + \theta_2 + \theta_3 = 180°$
$\theta_4 = \theta_1 + \theta_2$

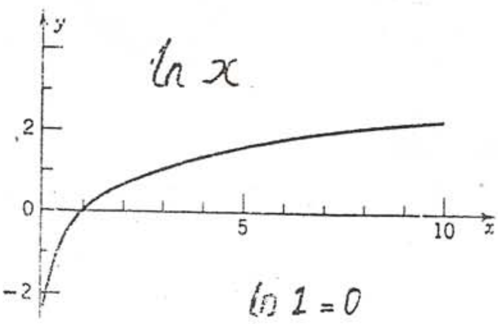


LOGARITHM

$e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71353$

$e^x e^y = e^{x+y}, \qquad e^x/e^y = e^{x-y}, \qquad (e^x)^y = e^{xy}$



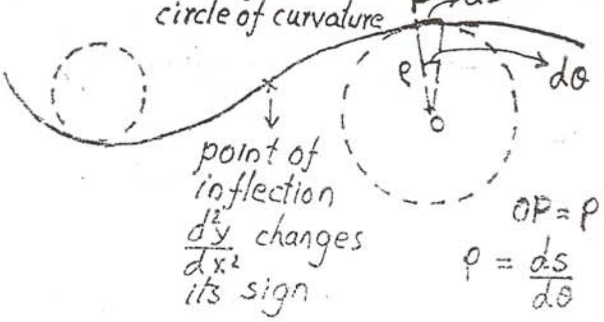


ln 1 = 0
ln 0 = −∞

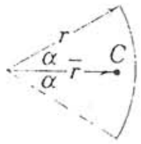
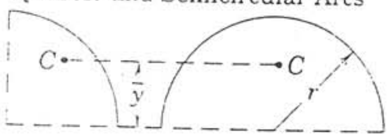
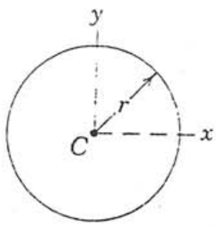
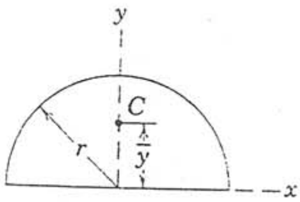
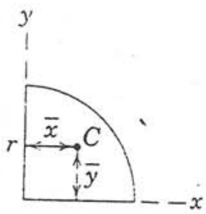
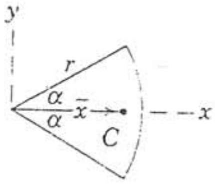
CURVATURE

$$\text{Radius of curvature} \begin{cases} \rho_{xy} = \dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} \\[2ex] \rho_{r\theta} = \dfrac{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}} \end{cases}$$

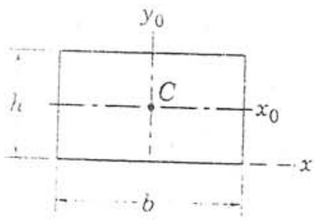
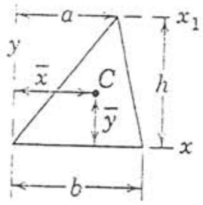
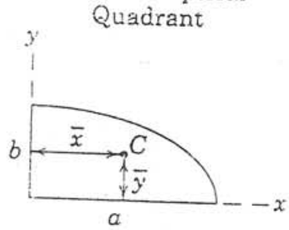
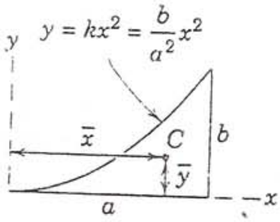
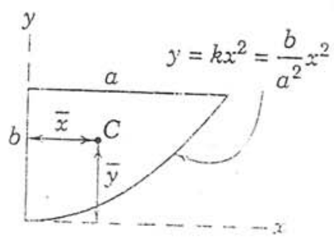
ds is an arc of the circle of curvature



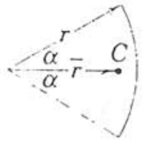
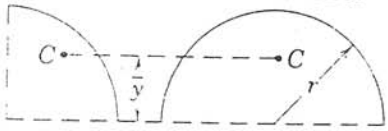
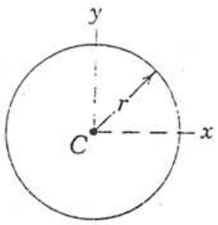
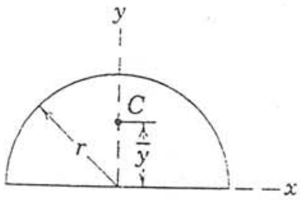
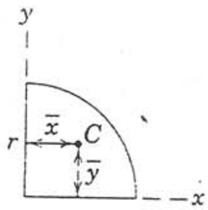
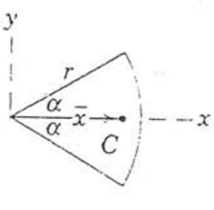
PROPERTIES OF PLANE FIGURES

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Arc Segment 	$\bar{r} = \dfrac{r \sin \alpha}{\alpha}$	—
Quarter and Semicircular Arcs 	$\bar{y} = \dfrac{2r}{\pi}$	—
Circular Area 	—	$I_x = I_y = \dfrac{\pi r^4}{4}$ $I_z = \dfrac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{8}$ $\bar{I}_x = \left(\dfrac{\pi}{8} - \dfrac{8}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \dfrac{2}{3} \dfrac{r \sin \alpha}{\alpha}$	$I_x = \dfrac{r^4}{4}(\alpha - \frac{1}{2} \sin 2\alpha)$ $I_y = \dfrac{r^4}{4}(\alpha + \frac{1}{2} \sin 2\alpha)$ $I_z = \frac{1}{2} r^4 \alpha$

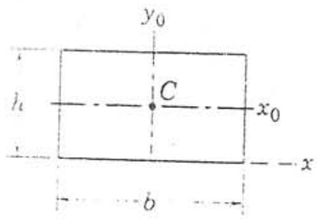
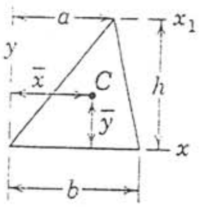
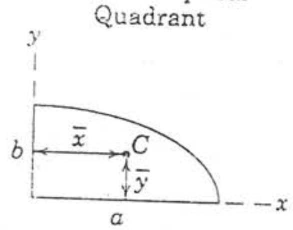
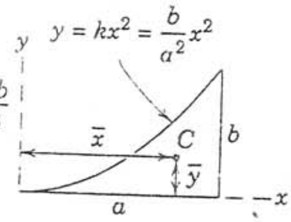
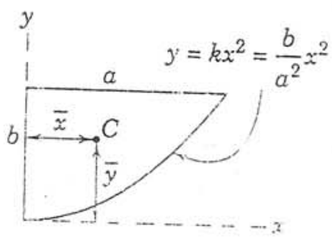
PROPERTIES OF PLANE FIGURES Continued

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Rectangular Area 	—	$I_x = \dfrac{bh^3}{3}$ $\bar{I}_x = \dfrac{bh^3}{12}$ $\bar{I}_z = \dfrac{bh}{12}(b^2 + h^2)$
Triangular Area 	$\bar{x} = \dfrac{a+b}{3}$ $\bar{y} = \dfrac{h}{3}$	$I_x = \dfrac{bh^3}{12}$ $\bar{I}_x = \dfrac{bh^3}{36}$ $I_{x_1} = \dfrac{bh^3}{4}$
Area of Elliptical Quadrant 	$\bar{x} = \dfrac{4a}{3\pi}$ $\bar{y} = \dfrac{4b}{3\pi}$	$I_x = \dfrac{\pi ab^3}{16}$, $\bar{I}_x = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right)ab^3$ $I_y = \dfrac{\pi a^3 b}{16}$, $\bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right)a^3 b$ $I_z = \dfrac{\pi ab}{16}(a^2 + b^2)$
Subparabolic Area Area $A = \dfrac{ab}{3}$ 	$\bar{x} = \dfrac{3a}{4}$ $\bar{y} = \dfrac{3b}{10}$	$I_x = \dfrac{ab^3}{21}$ $I_y = \dfrac{a^3 b}{5}$ $I_z = ab\left(\dfrac{a^2}{5} + \dfrac{b^2}{21}\right)$
Parabolic Area Area $A = \dfrac{2ab}{3}$ 	$\bar{x} = \dfrac{3a}{8}$ $\bar{y} = \dfrac{3b}{5}$	$I_x = \dfrac{2ab^3}{7}$ $I_y = \dfrac{2a^3 b}{15}$ $I_z = 2ab\left(\dfrac{a^2}{15} + \dfrac{b^2}{7}\right)$

PROPERTIES OF PLANE FIGURES

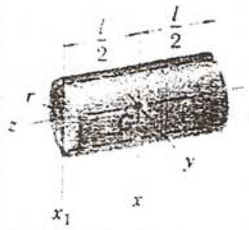
FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Arc Segment 	$\bar{r} = \dfrac{r \sin \alpha}{\alpha}$	—
Quarter and Semicircular Arcs 	$\bar{y} = \dfrac{2r}{\pi}$	—
Circular Area 	—	$I_x = I_y = \dfrac{\pi r^4}{4}$ $I_z = \dfrac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{8}$ $\bar{I}_x = \left(\dfrac{\pi}{8} - \dfrac{8}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \dfrac{2}{3} \dfrac{r \sin \alpha}{\alpha}$	$I_x = \dfrac{r^4}{4}(\alpha - \frac{1}{2} \sin 2\alpha)$ $I_y = \dfrac{r^4}{4}(\alpha + \frac{1}{2} \sin 2\alpha)$ $I_z = \frac{1}{2} r^4 \alpha$

PROPERTIES OF PLANE FIGURES Continued

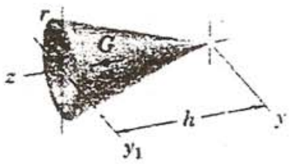
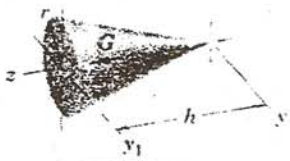
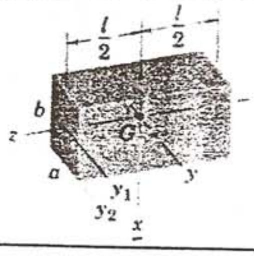
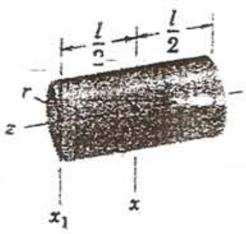
FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Rectangular Area 	—	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
Triangular Area 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$
Area of Elliptical Quadrant 	$\bar{x} = \frac{4a}{3\pi}$ $\bar{y} = \frac{4b}{3\pi}$	$I_x = \frac{\pi ab^3}{16}$, $\bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)ab^3$ $I_y = \frac{\pi a^3 b}{16}$, $\bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^3 b$ $I_z = \frac{\pi ab}{16}(a^2 + b^2)$
Subparabolic Area Area $A = \frac{ab}{3}$ 	$\bar{x} = \frac{3a}{4}$ $\bar{y} = \frac{3b}{10}$	$I_x = \frac{ab^3}{21}$ $I_y = \frac{a^3 b}{5}$ $I_z = ab\left(\frac{a^2}{5} + \frac{b^2}{21}\right)$
Parabolic Area Area $A = \frac{2ab}{3}$ 	$\bar{x} = \frac{3a}{8}$ $\bar{y} = \frac{3b}{5}$	$I_x = \frac{2ab^3}{7}$ $I_y = \frac{2a^3 b}{15}$ $I_z = 2ab\left(\frac{a^2}{15} + \frac{b^2}{7}\right)$

PROPERTIES OF HOMOGENEOUS SOLIDS

(m = mass of body shown)

BODY		MASS CENTER	MASS MOMENTS OF INERTIA
	Circular Cylindrical Shell	—	$I_{xx} = \frac{1}{2}mr^2 + \frac{1}{12}ml^2$ $I_{x_1x_1} = \frac{1}{2}mr^2 + \frac{1}{3}ml^2$ $I_{zz} = mr^2$
	Spherical Shell	—	$I_{zz} = \frac{2}{3}mr^2$
	Hemispherical Shell	$\bar{x} = \frac{r}{2}$	$I_{xx} = I_{yy} = I_{zz} = \frac{2}{3}mr^2$ $\bar{I}_{yy} = \bar{I}_{zz} = \frac{5}{12}mr^2$
	Sphere	—	$I_{zz} = \frac{2}{5}mr^2$
	Hemisphere	$\bar{x} = \frac{3r}{8}$	$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$ $\bar{I}_{yy} = \bar{I}_{zz} = \frac{83}{320}mr^2$
	Uniform Slender Rod	—	$I_{yy} = \frac{1}{12}ml^2$ $I_{y_1y_1} = \frac{1}{3}ml^2$

PROPERTIES OF HOMOGENEOUS SOLIDS (Continued)

BODY		MASS CENTER	MASS MOMENTS OF INERTIA
	Quarter-Circular Rod	$\bar{x} = \bar{y}$ $= \dfrac{2r}{\pi}$	$I_{xx} = I_{yy} = \frac{1}{2}mr^2$ $I_{zz} = mr^2$
	Conical Shell	$\bar{z} = \dfrac{2h}{3}$	$I_{yy} = \frac{1}{4}mr^2 + \frac{1}{2}mh^2$ $I_{y_1y_1} = \frac{1}{4}mr^2 + \frac{1}{6}mh^2$ $I_{zz} = \frac{1}{2}mr^2$ $\bar{I}_{yy} = \frac{1}{4}mr^2 + \frac{1}{18}mh^2$
	Right-Circular Cone	$\bar{z} = \dfrac{3h}{4}$	$I_{yy} = \frac{3}{20}mr^2 + \frac{3}{5}mh^2$ $I_{y_1y_1} = \frac{3}{20}mr^2 + \frac{1}{10}mh^2$ $I_{zz} = \frac{3}{10}mr^2$ $\bar{I}_{yy} = \frac{3}{20}mr^2 + \frac{3}{80}mh^2$
	Rectangular Parallelepiped	—	$I_{xx} = \frac{1}{12}m(a^2 + l^2)$ $I_{yy} = \frac{1}{12}m(b^2 + l^2)$ $I_{zz} = \frac{1}{12}m(a^2 + b^2)$ $I_{y_1y_1} = \frac{1}{12}mb^2 + \frac{1}{3}ml^2$ $I_{y_2y_2} = \frac{1}{3}m(b^2 + l^2)$
	Circular Cylinder	—	$I_{xx} = \frac{1}{4}mr^2 + \frac{1}{12}ml^2$ $I_{x_1x_1} = \frac{1}{4}mr^2 + \frac{1}{3}ml^2$ $I_{zz} = \frac{1}{2}mr^2$

VECTOR OPERATIONS

1. ***Notation.*** Vector quantities are printed in boldface type, and scalar quantities appear in lightface italic type. Thus, the vector quantity **V** has a scalar magnitude V. In longhand work vector quantities should always be consistently indicated by a symbol such as $\underline{V}$ or $\vec{V}$ to distinguish them from scalar quantities.

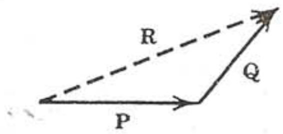
2. ***Addition***

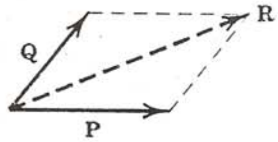
   Triangle addition $\mathbf{P} + \mathbf{Q} = \mathbf{R}$

   Parallelogram addition $\mathbf{P} + \mathbf{Q} = \mathbf{R}$

   Commutative law $\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P}$

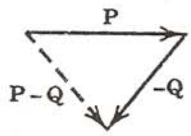
   Associative law $\mathbf{P} + (\mathbf{Q} + \mathbf{R}) = (\mathbf{P} + \mathbf{Q}) + \mathbf{R}$





3. ***Subtraction***

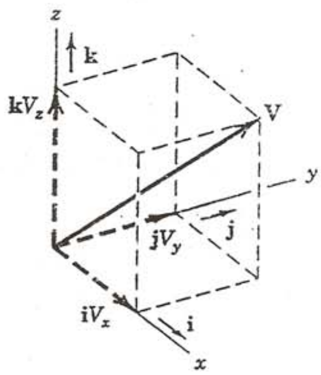
$$\mathbf{P} - \mathbf{Q} = \mathbf{P} + (-\mathbf{Q})$$



4. ***Unit vectors*** **i, j, k**

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$$

where $|\mathbf{V}| = V = \sqrt{V_x^{\,2} + V_y^{\,2} + V_z^{\,2}}$



5. ***Direction cosines*** l, m, n are the cosines of the angles between **V** and the x-, y-, z-axes. Thus,

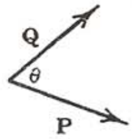
$$l = V_x/V \qquad m = V_y/V \qquad n = V_z/V$$

so that $\mathbf{V} = V(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$

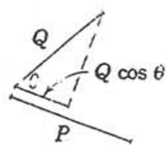
and $l^2 + m^2 + n^2 = 1$

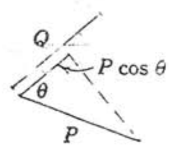
6. ***Dot or scalar product***

$$\mathbf{P}\cdot\mathbf{Q} = PQ\cos\theta$$



This product may be viewed as the magnitude of **P** multiplied by the component $Q\cos\theta$ of **Q** in the direction of **P**, or as the magnitude of **Q** multiplied by the component $P\cos\theta$ of **P** in the direction of **Q**.





Commutative law $\mathbf{P}\cdot\mathbf{Q} = \mathbf{Q}\cdot\mathbf{P}$

From the definition of the dot product

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0$$

$$\begin{aligned}\mathbf{P}\cdot\mathbf{Q} &= (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k})\cdot(Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k}) \\ &= P_xQ_x + P_yQ_y + P_zQ_z\end{aligned}$$

$$\mathbf{P}\cdot\mathbf{P} = P_x^{\,2} + P_y^{\,2} + P_z^{\,2}$$

It follows from the definition of the dot product that two vectors $\mathbf{P}$ and $\mathbf{Q}$ are perpendicular when their dot product vanishes, $\mathbf{P}\cdot\mathbf{Q} = 0$.

The angle θ between two vectors $\mathbf{P}_1$ and $\mathbf{P}_2$ may be found from their dot product expression $\mathbf{P}_1\cdot\mathbf{P}_2 = P_1P_2\cos\theta$, which gives

$$\cos\theta = \frac{\mathbf{P}_1\cdot\mathbf{P}_2}{P_1P_2} = \frac{P_{1_x}P_{2_x} + P_{1_y}P_{2_y} + P_{1_z}P_{2_z}}{P_1P_2} = l_1l_2 + m_1m_2 + n_1n_2$$

where l, m, n stand for the respective direction cosines of the vectors. It is also observed that two vectors are perpendicular to each other when their direction cosines obey the relation $l_1l_2 + m_1m_2 + n_1n_2 = 0$.

$$\text{Distributive law} \qquad \mathbf{P}\cdot(\mathbf{Q}+\mathbf{R}) = \mathbf{P}\cdot\mathbf{Q} + \mathbf{P}\cdot\mathbf{R}$$

7. ***Cross or vector product.*** The cross product $\mathbf{P}\times\mathbf{Q}$ of the two vectors $\mathbf{P}$ and $\mathbf{Q}$ is defined as a vector with a magnitude

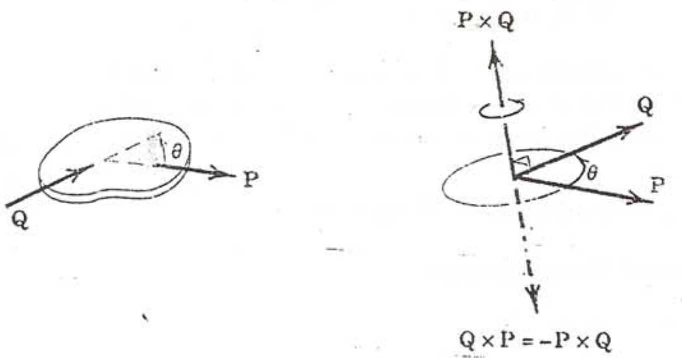
$$|\mathbf{P}\times\mathbf{Q}| = PQ\sin\theta$$

and a direction specified by the right-hand rule as shown. Reversing the vector order and using the right-hand rule give $\mathbf{Q}\times\mathbf{P} = -\mathbf{P}\times\mathbf{Q}$.

$$\text{Distributive law} \qquad \mathbf{P}\times(\mathbf{Q}+\mathbf{R}) = \mathbf{P}\times\mathbf{Q} + \mathbf{P}\times\mathbf{R}$$

From the definition of the cross product, using a *right-handed coordinate system*, we get

$$\mathbf{i}\times\mathbf{j} = \mathbf{k} \qquad \mathbf{j}\times\mathbf{k} = \mathbf{i} \qquad \mathbf{k}\times\mathbf{i} = \mathbf{j}$$
$$\mathbf{j}\times\mathbf{i} = -\mathbf{k} \qquad \mathbf{k}\times\mathbf{j} = -\mathbf{i} \qquad \mathbf{i}\times\mathbf{k} = -\mathbf{j}$$
$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$$



With the aid of these identities and the distributive law, the vector product may be written

$$\begin{aligned}\mathbf{P}\times\mathbf{Q} &= (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k})\times(Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k})\\ &= (P_yQ_z - P_zQ_y)\mathbf{i} + (P_zQ_x - P_xQ_z)\mathbf{j} + (P_xQ_y - P_yQ_x)\mathbf{k}\end{aligned}$$

The cross product may also be expressed by the determinant

$$\mathbf{P}\times\mathbf{Q} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ P_x & P_y & P_z\\ Q_x & Q_y & Q_z\end{vmatrix}$$

8. *Additional relations*

Triple scalar product $(\mathbf{P} \times \mathbf{Q}) \cdot \mathbf{R} = \mathbf{R} \cdot (\mathbf{P} \times \mathbf{Q})$. The dot and cross may be interchanged as long as the order of the vectors is maintained. Parentheses are unnecessary since $\mathbf{P} \times (\mathbf{Q} \cdot \mathbf{R})$ is meaningless because a vector $\mathbf{P}$ cannot be crossed into a scalar $\mathbf{Q} \cdot \mathbf{R}$. Thus, the expression may be written

$$\mathbf{P} \times \mathbf{Q} \cdot \mathbf{R} = \mathbf{P} \cdot \mathbf{Q} \times \mathbf{R}$$

The triple scalar product has the determinant expansion

$$\mathbf{P} \times \mathbf{Q} \cdot \mathbf{R} = \begin{vmatrix} P_x & P_y & P_z \\ Q_x & Q_y & Q_z \\ R_x & R_y & R_z \end{vmatrix}$$

Triple vector product $(\mathbf{P} \times \mathbf{Q}) \times \mathbf{R} = -\mathbf{R} \times (\mathbf{P} \times \mathbf{Q}) = \mathbf{R} \times (\mathbf{Q} \times \mathbf{P})$. Here we note that the parentheses must be used since an expression $\mathbf{P} \times \mathbf{Q} \times \mathbf{R}$ would be ambiguous because it would not identify the vector to be crossed. It may be shown that the triple vector product is equivalent to

$$(\mathbf{P} \times \mathbf{Q}) \times \mathbf{R} = \mathbf{R} \cdot \mathbf{P}\mathbf{Q} - \mathbf{R} \cdot \mathbf{Q}\mathbf{P}$$

or

$$\mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) = \mathbf{P} \cdot \mathbf{R}\mathbf{Q} - \mathbf{P} \cdot \mathbf{Q}\mathbf{R}$$

The first term in the first expression, for example, is the dot product $\mathbf{R} \cdot \mathbf{P}$, a scalar, multiplied by the vector $\mathbf{Q}$.

9. ***Derivatives of vectors*** obey the same rules as they do for scalars.

$$\frac{d\mathbf{P}}{dt} = \dot{\mathbf{P}} = \dot{P}_x\mathbf{i} + \dot{P}_y\mathbf{j} + \dot{P}_z\mathbf{k}$$

$$\frac{d(\mathbf{P}u)}{dt} = \mathbf{P}\dot{u} + \dot{\mathbf{P}}u$$

$$\frac{d(\mathbf{P} \cdot \mathbf{Q})}{dt} = \mathbf{P} \cdot \dot{\mathbf{Q}} + \dot{\mathbf{P}} \cdot \mathbf{Q}$$

$$\frac{d(\mathbf{P} \times \mathbf{Q})}{dt} = \mathbf{P} \times \dot{\mathbf{Q}} + \dot{\mathbf{P}} \times \mathbf{Q}$$

10. ***Integration of vectors.*** If $\mathbf{V}$ is a function of x, y, and z and an element of volume is $d\tau = dx\, dy\, dz$, the integral of $\mathbf{V}$ over the volume may be written as the vector sum of the three integrals of its components. Thus,

$$\int \mathbf{V}\, d\tau = \mathbf{i} \int V_x\, d\tau + \mathbf{j} \int V_y\, d\tau + \mathbf{k} \int V_z\, d\tau$$

SOLAR SYSTEM CONSTANTS

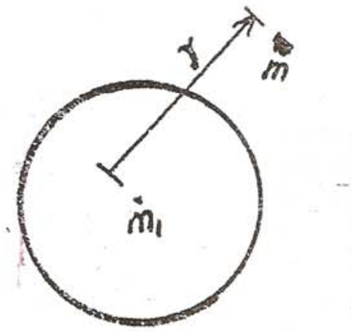
Universal gravitational constant	$G = 6.673(10^{-11})\ m^3/(kg \cdot s^2)$
	$= 3.439(10^{-8})\ ft^4/(lbf \cdot s^4)$
Mass of Earth	$m_e = 5.976(10^{24})$ kg
	$= 4.095(10^{23})\ lbf \cdot s^2/ft$
Period of Earth's rotation (1 sidereal day)	= 23 h 56 min 4 s
	= 23.9344 h
Angular velocity of Earth	$\omega = 0.7292(10^{-4})$ rad/s
Mean angular velocity of Earth–Sun line	$\omega' = 0.1991(10^{-6})$ rad/s
Mean velocity of Earth's center about Sun	= 107 200 km/h
	= 66,610 mi/h

BODY	MEAN DISTANCE TO SUN km (mi)	ECCENTRICITY OF ORBIT e	PERIOD OF ORBIT solar days	MEAN DIAMETER km (mi)	MASS RELATIVE TO EARTH	SURFACE GRAVITATIONAL ACCELERATION m/s^2 (ft/s^2)	ESCAPE VELOCITY km/s (mi/s)
Sun	—	—	—	1 392 000 (865 000)	333 000	274 (898)	616 (383)
Moon	384 398* (238 854)*	0.055	27.32	3 476 (2 160)	0.0123	1.62 (5.32)	2.37 (1.47)
Mercury	57.3×10^6 (35.6×10^6)	0.206	87.97	5 000 (3 100)	0.054	3.47 (11.4)	4.17 (2.59)
Venus	108×10^6 (67.2×10^6)	0.0068	224.70	1[illegible]400 ([illegible]700)	0.815	8.44 (27.7)	10.24 (6.36)
Earth	149.6×10^6 (92.96×10^6)	0.0167	365.26	12 742† (7 918)†	1.000	9.821‡ (32.22)‡	11.18 (6.95)
Mars	227.9×10^6 (141.6×10^6)	0.093	686.98	6 788 (4 218)	0.107	3.73 (12.3)	5.03 (3.13)

* Mean distance to Earth (center-to-center)

† Diameter of sphere of equal volume, based on a spheroidal Earth with a polar diameter of 12 714 km (7900 mi) and an equatorial diameter of 12 756 km (7926 mi)

‡ For nonrotating spherical Earth, equivalent to absolute value at sea level and latitude 37.5°



The evolution of g (acceleration due to gravity)

Newton's law of gravitation states $F = Gm_1m_2/r^2$. Considering m_1 as the mass of earth and m as the mass of any body at a distance r from the centre of earth and noting that the Universal gravitation constant, $G = 6.673 \times 10^{-11}\ m^3/kgs^2$ and $m_1 = 5.976 \times 10^{24}$ kg

$F = (6.673 \times 10^{-11})(5.976 \times 10^{24})\ m/r^2$ simplifies to

$F = m\,(3.9877848 \times 10^{14})/r^2$

If the mass m is on or near the surface of earth, r can be approximately taken as r = 6376000 m. When this value is plugged into F,

$F = m \times 9.809234047$

The highlighted number is our 'pet' $g = 9.81\ m/s^2$.