Numerical Methods

Notes by-

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* Finding Roots of Equation by - ① Bisection Mtd.
② Newton-Rapson Mtd.

There are only a few types of equations which can be solved by simple direct methods, eg. linear eq^n $ax+b~~x~~=0$ or the quadratic eq^n $ax^2+bx+c=0$.

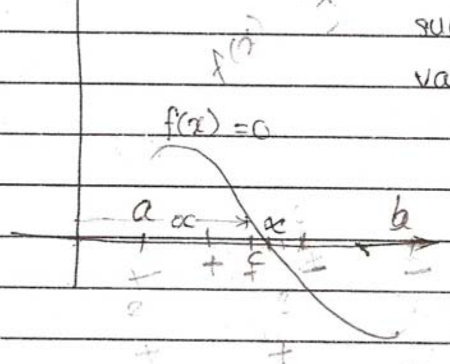
The majority of eq^n encountered in Engg. applications are of types which do not posses a direct solution procedure. If a small change is made in the above eq^n i.e.

$ax^{2.3}+bx+c=0$, we find that this is not so ~~re~~ easily solvable. Such types of eq^n are called as "Transcendental Equations" & they contain functions which are not all algebraic.
The commonest transcendental functions are LIATE i.e. Trignometric, exponential & logarithmic function.

When direct method of sol^n are not available, we use "iterative technique". In an iterative technique, an initial estimate ~~are~~ of the solution is somehow obtained & then further improvements to this estimate are made. We can make the error as small as possible, but we cannot get the exact solution i.e. open end solution.

Iterative Methods (~~Iterative~~) for finding roots of transcendental eq^n :-
1) Bisection Method
2) Secant Method
3) Regula - Falsi Method
4) Newton - Rapson method.

1) Bisection Method :-



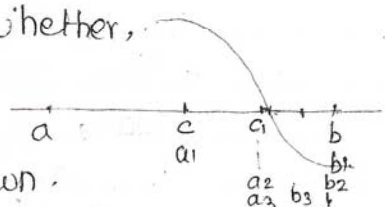
This method ~~depend~~ based on successive application of intermediate value Theorem.

Theorem :- If a f^n is continuous in $[a,b]$ & if $f(a)$ & $f(b)$ have opposite signs, then eq^n $f(x)=0$ has at least one real root (α) in the interval $[a,b]$

i.e. $f(a)\cdot f(b)$ is $-ve$, $(f(a)\cdot f(b) < 0)$
then $f(x)=0$ has at least one real root α in $[a,b]$

Procedure:-

1) Let $f(x)$ be continuous in $[a,b]$ & $f(a)\cdot f(b) = -ve$ then $f(x)=0$ has a root in $[a,b]$
2) - Find middle point of $[a,b]$ i.e. $\left(\frac{a+b}{2}\right)$ & check whether,
   $f(a)\cdot f(c) = -ve$ or $f(b)\cdot f(c) = -ve$.
   By constructing interval $[a,c]$ & $[c,b]$
   if $f(b)\cdot f(c) = -ve$, Root lies betⁿ $[c,b]$ as shown.
   Thus we have reduced interval $[a,b]$ to half interval say $[a_1,b_1]$



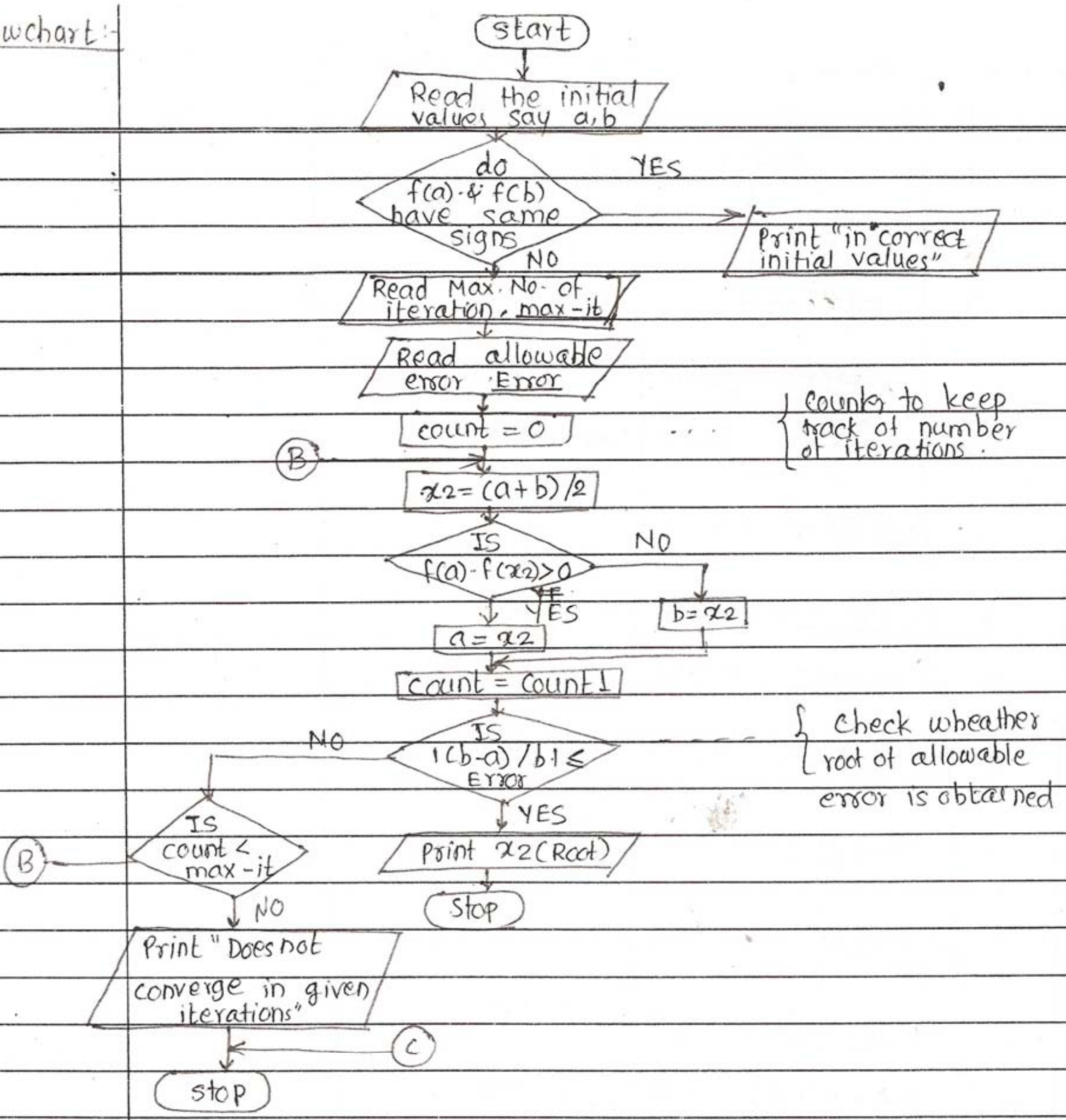
3] Now we apply step (2) to $[a_1,b_1]$ & the interval $[a_1,b_1]$ is halved to $[a_2,b_2]$ & the process is repeated.
4] At some stage, either we get an exact root or an infinite sequence of intervals $[a_1,b_1]$, $[a_2,b_2]$, ... $[a_n,b_n]$ & we have,

$$f(a_n)\cdot f(b_n) < 0$$
$$\therefore b_n - a_n = \frac{1}{2^n}(b-a)$$

* Algorithm:-

step 1: Read two points $[a,b]$ such that $f(a)$ & $f(b)$ are of opposite sign.
step 2: Find mid point of $[a,b]$ i.e. bisect them i.e. $x_2 = (a+b)/2$
step 3: Check the position of root:
  i) If $f(a)\cdot f(x_2)$ is $-ve$, Root lies betⁿ a & x_2 $\therefore b = x_2$
  ii) If $f(a)\cdot f(x)$ is $+ve$, Root lies betⁿ b & x_2 $\therefore a = x_2$
Step 4: Repeat step 2 & 3 until the root of reqd. precision is obtained.
Step 5: Stop.

* Flowchart:-



Pro:- Find the root of eq^n $x^3 + x^2 - 1 = 0$.

Sol^n:- let $f(x) = x^3 + x^2 - 1$

First we obtain min. values of a & b, such that they have opposite signs.

$\therefore f(1) = +1$ $\quad$ $\therefore$ ~~$f(0.5) = +0.75$~~ $\quad$ $f(-1) = -1$

$f(2) = +11$ $\quad$ ~~$f(0.25) = 0.7548$~~

$\therefore$ We assume that root lies bet^n $[0.25, -1]$

n	x	$f(x) = x^3+x^2-1$
1	$\cancel{0.25}$ 1 -1	$+1$ -1
2	$\frac{+1-1}{2} = 0$	-1
3	$\frac{+0+1}{2} = 0.5$	-0.625
4	$\frac{-0.\cancel{25}5+1}{2} = \cancel{0.375}\ 0.75$	$\cancel{-0.9583}$ -0.0156
5	$\frac{\cancel{-0.9583}\ +0.75+1}{2} = \cancel{0.0209}\ 0.875$	$+0.4355$
6	$\frac{0.875+0.75}{2} = 0.8125$	$+0.1965$
7	$\frac{0.8125+0.75}{2} = 0.7813$	$+0.0872$
8	$\frac{0.7813+0.75}{2} = 0.7657$	$+0.0351$
9	$\frac{0.7657+0.75}{2} = 0.7579$	$+0.0096$
10	$\frac{0.7579+0.75}{2} = 0.7540$	-0.003
11	$\frac{0.7540+0.7579}{2} = 0.7560$	$+0.0035$
12	$\frac{0.7560+0.7540}{2} = 0.755$	$+0.0004$
13	$\frac{0.755+0.7540}{2} = 0.7545$	-0.0012
14	$\frac{0.7545+0.755}{2} = 0.7548$	-0.0004
15	$\frac{0.7548+0.755}{2} = 0.7549$	$+0.0001$
16.	$\frac{0.7549+0.7548}{2} = 0.75485$	0

$\therefore$ Root of eqⁿ $(x^3+x^2-1 = 0)$ is $\underline{\underline{0.75485}}$

& this root is obtained at 16 iteration with 4 decimal

Pro: Obtain the real root of $x^3-x^2-x-1=0$ lying betⁿ 1.7 & 1.9, correct up to 4 places of decimal by bisection method

n	x	$f(x)=x^3-x^2-x-1$
0	1.7	−0.6770
	1.9	+0.3490
1	$\frac{1.7+1.9}{2}=1.8000$	−0.2080
2	$\frac{1.800+1.9}{2}=1.8500$	+0.0591
3	$\frac{1.8500+1.8000}{2}=1.8250$	−0.0772
4.	$\frac{1.8250+1.8500}{2}=1.8375$	−0.0098
5.	$\frac{1.8375+1.8500}{2}=1.8438$	+0.0265
6	$\frac{1.8438+1.8375}{2}=1.8407$	+0.0075
7	$\frac{1.8407+1.8375}{2}=1.8391$	−0.001
8	$\frac{1.8391+1.8407}{2}=1.8399$	+0.0034
9	$\frac{1.8399+1.8391}{2}=1.8395$	+0.0012
10	$\frac{1.8395+1.8391}{2}=1.8393$	+0.0001
11	$\frac{1.8393+1.8391}{2}=1.8392$	−0.0005
12	$\frac{1.8392+1.8393}{2}=1.8393$	−0.0002

} ∴ Root lies betⁿ [1.7, 1.9]

Save this eqⁿ in calcⁱ

$Ans^3-Ans^2-Ans-1$

$\boxed{\text{New value}}$ $\boxed{=}$ $\boxed{\text{Back}}$ Eqⁿ $\boxed{=}$ Final Ans.

∴ Root of eqⁿ $x^3-x^2-x-1=0$ is 1.8393 obtained at 12ᵗʰ iteration & correct up to 4 decimal places.

Pro: Perform 5 iteratⁿ of the bisection method to obtain the smallest positive root of the eqⁿ $f(x) = x^3 - 5x + 1$.

Solⁿ:-

n	x	$f(x) = x^3 - 5x + 1$
	0	+1 } ∴ Root lies betⁿ [0, 1]
	1	−3
1	$\frac{1+0}{2} = 0.5$	−1.3750
2	$\frac{0.5+0}{2} = 0.25$	−0.2344
3.	$\frac{0.25+0}{2} = 0.125$	+0.3770
4	$\frac{0.125+0.25}{2} = 0.1875$	+0.0691
5	$\frac{0.1875+0.25}{2} = 0.2188$	−0.0833

∴ At 5th iteration, we obtain root of eqⁿ $x^3 - 5x + 1$ is <u>0.2188</u>

Pro: Using bisection mtd. find the root of the eqⁿ. $x^3 - 1.8x^2 - 10x + 17 = 0$ that lies in the interval [1, 2] at the end of 5th iteration.

n	x	$f(x) = x^3 - 1.8x^2 - 10x + 17$
	1	+6.2000 } Root lies betⁿ [1, 2]
	2	−2.2000
1	$\frac{1+2}{2} = 1.5000$	+1.3250
2.	$\frac{1.5+2}{2} = 1.7500$	−0.6531
3	$\frac{1.75+1.5}{2} = 1.6250$	+0.2879
4	$\frac{1.6250+1.7500}{2} = 1.6875$	−0.1954
5	$\frac{1.6875+1.6250}{2} = 1.6563$	+0.0432

∴ Root of eqⁿ $x^3 - 1.8x^2 - 10x + 17 = 0$, ~~is~~ obtained at 5th iteratⁿ = <u>1.6563</u>

Pro: Find the root of eq^n $3\sin x + e^x - 5x^2 + 3 = 0$ up to 4 sign. digits.

Make in Rad

n	x	$f(x) = 3\sin x + e^x - 5x^2 + 3$
	0	+4
	+1	+0.7706
	−1	+8.3155
	+2	~~+9.4938~~ −9.5062
	~~3~~	~~+8.2425~~
	~~−2~~	
1	$\frac{+1+2}{2} = 1.5$	−3.6898
2	$\frac{1.5+1}{2} = 1.2500$	−1.2567
3	$\frac{1.25+1}{2} = 1.1250$	−0.1890
4	$\frac{1.125+1}{2} = 1.0625$	+0.3047
5	$\frac{1.0625+1.125}{2} = 1.0938$	+0.0613
6	$\frac{1.0938+1.125}{2} = 1.1094$	−0.0632
7	$\frac{1.1094+1.0938}{2} = 1.1016$	−0.0010
8	$\frac{1.1016+1.0938}{2} = 1.0977$	+0.0300
9	$\frac{1.0977+1.1016}{2} = 1.0997$	+0.0145
10	$\frac{1.0997+1.1016}{2} = 1.1007$	+0.0066
11	$\frac{1.1007+1.1016}{2} = 1.1012$	+0.0026
12	$\frac{1.1012+1.1016}{2} = 1.1014$	+0.0006
13	$\frac{1.1014+1.1016}{2} = 1.1015$	−0.0002

Root lies bet^n [+1, +2]

$\therefore$ Root of eq^n $3\sin x + e^x - 5x^2 + 3 = 0$, obtained is 1.1015 correct up to 4 dc. place

(2) Secant Method :-

$$\boxed{x_{r+1} = x_r - \frac{x_r - x_{r-1}}{f(x_r) - f(x_{r-1})} \cdot f(x_r)}$$

From above equation, consider any two roots $[x_0, x_1]$ betⁿ which fⁿ lies. find x_2
Substitute value of x_2 in the given eqⁿ to get −ve value.
~~Repeat~~ Now select roots as $[x_1, x_2]$ find x_3 & repeat the procedure until the root of eqⁿ is obtained up to reqd. decimal.

eg :- Use secant method, to find root of $f(x) = x^3 - 5x - 7 = 0$ correct up to 3 places of decimal.

solⁿ :- First we find two numbers betⁿ which function lies.

n	x	$f(x) = x^3 - 5x - 7$
	1	−11.00
	$\boxed{x_0 = 2}$	−9.00
	−1	−3
	−2	−107
	$\boxed{x_1 = 3}$	+5 ∴ Root lies betⁿ [2,3]
	∴ $x_0 = 2,\ x_1 = 3$	
1	$x_2 = x_{\cancel{0}} - \frac{(x_1 - x_0)}{[f(x_1) - f(x_0)]} \cdot f(x_1) = 2.6429$	−1.7547
2	$x_3 = x_2 - \frac{(x_2 - x_1)\cdot f(x_2)}{[f(x_2) - f(x_1)]}$ = (Root lies betⁿ [3, 2.6429])	
	$= 2.6429 - \frac{[(2.6429 - 3) \times (-1.7547)]}{[-1.7547 - (+5)]} = \cancel{2.1397}$ 2.7357	~~−7.9021~~ −0.2050
3	$x_2 = x_1 - \frac{(x_1 - x_0)\, f(x_1)}{[f(x_1) - f(x_0)]}$ ∴ (Root lies betⁿ) $[\underset{x_0}{2.7357}, \underset{x_1}{3}]$	
	~~(Root lies betⁿ [3, 2.1397]~~ ⇒ ~~$x_3 = 2.1397$~~ ~~$x_2 = 3$~~	~~−7.3752~~
	∴ $x_2 = 2.7461$	−0.0218
4.	∴ Root lies betⁿ [2.7461, 3]	
	∴ $x_2 = 2.7461 - \frac{(2.7461 - 3)(-0.0218)}{[-0.0218 - 5]} = 2.7472$	−0.0025
5.	∴ Root lies betⁿ [2.7472, 3]	
	$x_2 = \frac{2.7472 - (2.7472 - 3)(-0.0025)}{(-0.0025 - 5)} = \boxed{2.7473}$ Ans	−0.0003

Pro: Find a real root of $\cos x - xe^x = 0$ by secant method.

n	x	$f(x) = \cos x - xe^x$
	x_1 1	-1.7184
	x_0 0	$+1$
	$\therefore x_0 = 0,\ x_1 = 1$	
1	$x_2 = x_1 - \dfrac{(x_1 - x_0)\cdot f(x_1)}{[f(x_1) - f(x_0)]} = 0.3679$	$+0.4685$
2	Root lies betn $[\cancel{x_0}\ 0.3679, 1]$ $\therefore x_2 = 1 - \dfrac{(0.3679+1)(0.4685)}{(0.4685 + 1.7184)} = 0.8646$	-1.0527
3	Root lies betn $[\overset{x_0}{0.3679}, \overset{x_1}{0.8646}]$ $x_2 = 0.8646 - \dfrac{(0.8646 - 0.3679)(-1.0527)}{(-1.0527 - 0.4685)} = \cancel{1.6876}$ $= 0.5409$	~~-8.1240~~ $+0.0710$
4.	Root lies betn $(0.5409, 0.8646)$ $x_2 = 0.8646 - \dfrac{(0.8646 - 0.5409)(0.0710)}{(0.0710 + 1.0527)}$ $= 0.8441$	-0.9636
5.	Root lies betn $(0.5409, 0.8441)$ $\therefore x_2 = 0.8441 - \dfrac{(0.8441 - 0.5409)(-0.9636)}{(-0.9636 - 0.0710)}$ $= 0.5617$	$+0.0149$
6	Root lies betn $(0.5617, 0.8441)$ $x_2 = 0.8441 - \dfrac{(0.8441 - 0.5617)(-0.9636)}{(-0.9636 - 0.0149)}$ $= 0.5660$	$+0.0031$

correct

$\therefore$ Root of eqn is $\underline{\underline{0.566}}$

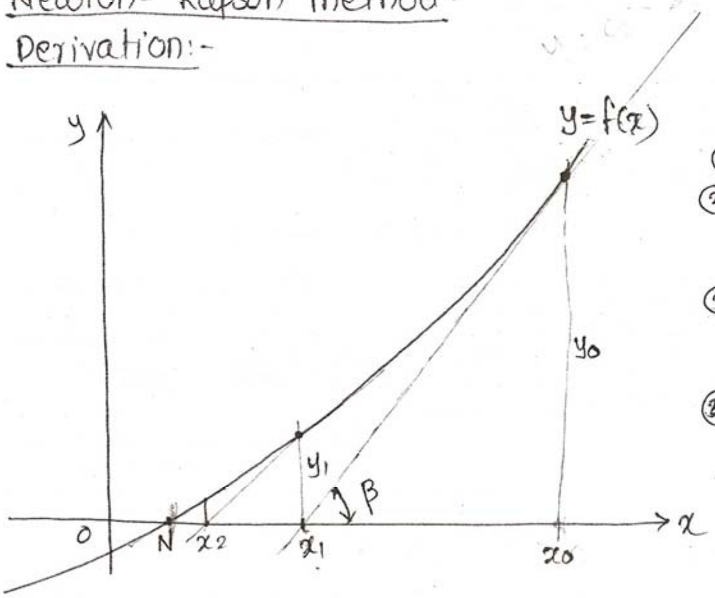
③ Regular Falsi or Method of False Position:-

$$\boxed{x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}}$$

Procedure is same as secant method.

④ Newton-Rapson method:-

Derivation:-



① select any value 'x_0'
② Find corresponding 'y_0' by substituting 'x_0' in the eq^n $y_0 = f(x_0)$
③ Draw tangent at $y_0 = f(x_0)$ which cuts x axis at x_1.
④ consider $x_1 = x_0$ & Repeat step ② & ③ till we reach point 'N' which is the root of eq^n as shown in fig.

Newton-Rapson method is generally more expensive per iteration, but it converges more rapidly & hence it requires fewer iterations to attain a given desired accuracy.

from fig, $\tan\beta = \dfrac{y_0}{(x_0 - x_1)} = \dfrac{f(x_0)}{x_0 - x_1} = f'(x_0)$ i.e. slope of tangent.

$$\therefore \frac{f(x_0)}{f'(x_0)} = x_0 - x_1$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$\therefore$ In general, $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$

$$x_{n+1} = Ans - \frac{f(Ans)}{f'(Ans)}$$

Algorithm :-

step 1) Read the initial guess value of root, say "x_0"

step 2) Read the allowable error, Say "Err".

step 3) Read the max. No. of iterations say "max-it"

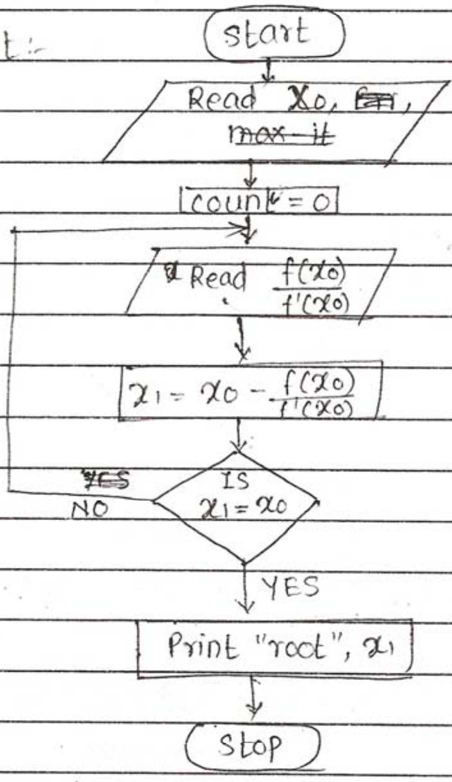
step 4) Read the slope value, say "$\frac{f(x_0)}{f'(x_0)}$"

step 5) Find the new guess of root, say, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

step 6) Repeat the step 4 & 5 by considering $x_1 = x_0$ till, $x_n = x_{n-1}$

step 7) Stop.

* Flow Chart :-



Pro :- By using Newton Rapson method, find the root of $x^4 - x - 10 = 0$ which is near to $x = 2$.

Sol^n :-

n	x	$f(x_0) = x^4 - x - 10$
	$x_0 = 2$	$\therefore f'(x_0) = 4x^3 - 1$
1	$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{[2^4 - 2 - 10]}{[4 \times 2^3 - 1]} = 1.8710$	
2	$x_2 = 1.8558$	
3	$x_3 = 1.8556$	
4	$\boxed{x_4 = 1.8556}$	$\therefore$ Root of eq^n is 1.8556

sin - cosine
cos - sec
tan - cot

Pro:- Using Newtons method, find the smallest positive root of the eq^n $\tan x = x$, correct up to ~~4~~ 4 decimals.

Sol^n:-

n $\quad f(x_0) = \tan x - x$, $f'(x_0) = \sec^2 x - 1$

$= \frac{1}{\cos^2 x} - 1$

$= \frac{1 - \cos^2 x}{\cos^2 x}$

1 $\quad$ Assume $x_0 = \pi/2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= \frac{\pi}{2} - \frac{[\tan(\pi/2) - \pi/2]}{\left(\frac{1 - \cos^2(\pi/2)}{\cos^2(\pi/2)}\right)} =$$

This eq^n cannot be easily solved.

$\therefore \tan x = x$

$\therefore \frac{\sin x}{\cos x} = x$

$\therefore \sin x = x \cos x$

$\therefore f(x) = \sin x - x \cos x$

$f'(x) = \cos x - [x(-\sin x) + \cos x]$

$= \cos x + x \sin x - \cos x$

$= x \sin x$

$\therefore$

n $\quad f(x) = \sin x - x \cos x$

$f'(x) = x \sin x$

1 $\quad$ Assume $x = 3\pi/2$ Such that $f(3\pi/2)$ is closer to zero.

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{[\cancel{\sin}(3\pi/2) - \sin(3\pi/2) - (\frac{3\pi}{2})\cos\frac{3\pi}{2}]}{3\pi/2 \sin(3\pi/2)}$$

$= 4.7124$

2 $\quad x_2 = 4.5002$

3 $\quad x_3 = 4.4934$

4 $\quad x_4 = 4.4934$

$\therefore$ Smallest possible root of eq^n $\tan x = x$ is 4.4934 obtained at 4th iteration correct up to 4 significant digits.

Pro: Use Newton Rapson mtd. to compute root of the eqⁿ

$$x \log_{10} x - 1.2 = 0$$

Let $f(x) = x \log_{10} x - 1.2$

$\therefore f(0) = \infty$

$f(1) = -1.2$

$f(2) = -0.5979$

$f(3) = 0.2314.$

$\therefore$ Root lies betⁿ 2 & 3. & close to 3

$\therefore x_0 = 3.$

n

$$f(x) = x \log_{10} x - 1.2$$

$$\cancel{f'(x) = x \frac{1}{\log_{10} x} + \log_{10} x \cdot 1}$$

$$\therefore f(x) = \frac{x \cdot \log x}{\log 10} - 1.2$$

$$f'(x) = \frac{1}{\log 10}\left[x \cdot \frac{1}{x} + \log x\right] - 0$$

$$= \frac{1}{\log 10}\left[1 + \log x\right]$$

$\therefore$ 1 $\quad \cancel{x_1 = [3 \log_{10} 3 - 1.2]}$

$$\therefore x_1 = 3 - \frac{[3 \log_{10} 3 - 1.2]}{\left(\frac{1 + \log 3}{\log 10}\right)} = 2.8898$$

2	$x_2 = 2.8258$
3	$x_3 = 2.7891$
4	$x_4 = 2.7682$
5	$x_5 = 2.7562$
6	$x_6 = 2.7495$
7	$x_7 = 2.7456$
8	$x_8 = 2.7435$
9	$x_9 = 2.7422$
10	$x_{10} = 2.7416$
11	$x_{11} = 2.7412$
12	$x_{12} = 2.7409$
13	$x_{13} = 2.7408$
14	$x_{14} = 2.7407$
15	$x_{15} = 2.7407$

~~$\therefore$ solⁿ of e.~~

$\therefore$ Root of eqⁿ $x \log_{10} x - 1.2 = 0$ is $\underline{\underline{2.7407}}$

Pro:- Find the root of eq^n $xe^x-1=0$ using Newton Rapson mtd.

let, $f(x) = xe^x-1$

$f(0) = -1$

$f(1) = +1.7183$

$\therefore$ Root of eq^n lies bet^n 0 & 1.

consider $x_0 = 1$.

$f(x) = xe^x-1$

$f'(x) = x \cdot e^x + e^x$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= x_0 - \frac{x_0 \cdot e^{x_0} - 1}{x_0 \cdot e^{x_0} + e^{x_0}}$$

n	x
1	$x_1 = 1 - \frac{[1 \times e^1 - 1]}{[1 \times e^1 + e^1]} = 0.6839$
2	$x_2 = 0.5775$
3	$x_3 = 0.5672$
4	$x_4 = 0.5671$
5	$x_5 = 0.5671$ ... approx.

$\therefore$ Root of eq^n $xe^x-1=0$ is 0.5671 correct up to 4 significant digit obtained at 5^{th} iteration.

Pro: Use, Newton Rapson mtd. to obtain a root to 4 decimal places of the following eq^n, ~~xcosx~~ $x\sin x + \cos x = 0$

Sol^n:- $f(x) = \cancel{x\cos}\ x\sin x + \cos x$

$\therefore f(0) = 1$

$f(\pi/2) = +1.5708$

$f(\pi/4) = +1.2625$

$f(\pi) = -1$.

$\therefore$ Root lies bet^n 0 & π

$\therefore$

Consider $x_0 = 0$.

$f(x) = x\sin x + \cos x$

$f'(x) = x\cos x + \sin x + \cos x$

n	x
1	$x_1 = 0 - \frac{[0\sin 0 + \cos 0]}{[0\cos 0 + \sin 0 + \cos 0]} = -1.000$
2	$x_2 = 0.6421$
3	$x_3 = 0.0228$
4	$x_4 = -0.9342$
5	$x_5 = 0.8248$
6	$x_6 = 0.1737$
7	$x_7 = -0.5900$
8	$x_8 = 4.7848$
9	$x_9 = -3.3321$
10	$x_{10} = -2.6816$
11	$x_{11} = -2.9587$
12	$x_{12} = -2.7034$
13	$x_{13} = -2.9193$
14	$x_{14} = -2.7184$
15	$x_{15} = -2.8954$
16	$x_{16} = -2.7297$
	$x_{17} = -2.8786$
	$= -2.7388$
	-2.8660
	-2.7463
	-2.8561
	-2.7526
	-2.8481
	-2.7580
	-2.8415
	-2.7627
	-2.8359

very short Date: 4/02/05

so long process....

$\therefore$ Root of eq^n $x\sin x - \cos x = 0$ is -2.7984 obtained at ... very long iterat^n & correct up to 4 decimal places.

Pro:- $x^4 - x - 10$ $\qquad f'(x) = 4x^3 - 1$

$f(1) = -10$, $f(2) = 4$

$\therefore n = 0$ $\qquad f_1 = 2 - \frac{(2)^4 - 2 - 10}{4(2)^3 - 1}$

1	1.8710
2	1.8558
3	1.8556
4	1.8556 .. Ans.