Numerical Methods

Notes by-

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SOLUTION OF LINEAR ALGEBRAIC EQUATION.

If we have to solve 50 eq^n in 50 unknowns which may occur when dealing with space frames used in roof trusses, bridge trusses, etc, the evaluation is time consuming process. In such a situation a numerical approach is adopted which is purely mechanical in operation.

Thus matrix notation is convinient & powerful to represent the wide variety of problems.

Basic concepts :-

The matrix of order mxn (m rows & n column) is represented as,

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & \underline{x_{mn}} \end{bmatrix}_{m \times n}$$

Square Matrix :- If m=n i.e. No. of rows = No. of col^m such matrix is known as square matrix.

$[x_{11} \quad x_{12} \quad x_{13} \quad \cdots \quad x_{1n}]$ .... Row Vector

$$\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ \vdots \\ x_{m1} \end{bmatrix}$$ ... column Vector.

The elements whose row & col^m positions are same, are called diagonal elements, of a square matrix eg: $x_{11}, x_{22}, x_{33}, \ldots x_{nn}$

The sum of diagonal elements gives us the "TRACE" of matrix.

* Upper triangular Matrix :- If all the elements below the diagonal elements are zero, in case of square matrix, is known as upper triangular matrix. [i.e. $x_{ij} = 0$ for $i > j$]

eg:

$$X = \begin{bmatrix} 5_{11} & 8_{12} & 9_{13} \\ 0_{21} & 6_{22} & 2_{23} \\ 0_{31} & 0_{32} & 7_{33} \end{bmatrix}$$ Upper triangular matrix.

* Lower triangular Matrix :- A square matrix is said to be lower triangular matrix if all the elements above the diagonal elements are zero. [i.e. $x_{ij} = 0$ $j < i$]

eg:

$$X = \begin{bmatrix} 9_{11} & 0_{12} & 0_{13} \\ 5_{21} & 8_{22} & 0_{23} \\ 6 & 4 & 7 \end{bmatrix}$$

* Symmetrical matrix : $x_{ij} = x_{ji}$

eg:

$$X = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$$

* Skew Matrix :- $x_{ij} = -x_{ji}$

eg:-

$$X = \begin{bmatrix} 1 & 4 & 5 \\ -4 & 2 & 6 \\ -5 & -6 & 3 \end{bmatrix}$$

* Minor of ~~me~~ element :-

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}.$$

Minor of $x_{11} = \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix}$; Minor of $x_{33} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$

Minor of $x_{22} = \begin{vmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{vmatrix}$ & so. on...

* Cofactor of $x_{ij} = (-1)^{i+j} \cdot M_{ij}$ Where M_{ij} = Minor of x_{ij}

* To find determinant of $A = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$

$\therefore |A| = 5 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - (-2) \begin{vmatrix} -2 & 1 \\ 4 & 0 \end{vmatrix} + 4 \begin{vmatrix} -2 & 1 \\ 4 & 1 \end{vmatrix} =$

$= 5(1\times0 - 1\times1) + 2(-2\times0 - 4\times1) + 4(-2\times1 - 4\times1)$

$\boxed{|A| = -37}$

* To find inverse of matrix : $A = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$

$[A]^{-1} = |A| \times [\text{Cofactor } (A)]$

$\therefore |A| = -37$; $\boxed{\text{Cofactor } x_{ij} = (-1)^{i+j} \cdot M_{ij}}$

cofactor of $5 = 1\times0 - 1\times1 = -1 \times (-1)^{i+j} = -1 \times (-1)^{1+1} = -1$

$-2 = 0 - 4 = -4 \times (-1)^{1+2} = 4$

$+4 = -2 - 4 = -6 \times (-1)^{1+3} = -6$

$-2 = 0 - 4 = -4 \times (-1)^{2+1} = 4$

$1 = 0 - 16 = -16 \times (-1)^{2+2} = -16$

$1 = 5 + 8 = 13 \times (-1)^{2+3} = -13$

$4 = -2 - 4 = -6 \times (-1)^{3+1} = -6$

$1 = 5 + 8 = 13 \times (-1)^{3+2} = -13$

$0 = 5 - 4 = 1 \times (-1)^{3+3} = 1$

$\therefore [A]^{-1} = \frac{1}{-37} \begin{bmatrix} -1 & +4 & -6 \\ +4 & -16 & -13 \\ -6 & -13 & 1 \end{bmatrix} = \frac{1}{37} \begin{bmatrix} 1 & -4 & 6 \\ -4 & 16 & +13 \\ 6 & +13 & -1 \end{bmatrix}$

* Rank of Matrix :-

A matrix is said to be of rank 'r' if,

i) It has ~~tea~~ at least one nonzero minor of order 'r'.

ii) Every minor of order higher than 'r' vanish (i.e. their value becomes zero)

$\frac{N}{10}$

Solve $5x+3y+7z=4$

$3x+26y+2z=9$

$7x+2y+11z=5$ by matrix inversion.

→ We write the eq^n in the form of matrix eq^n,

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

i.e. $A \cdot X = D$ i.e. $A^{-1} \cdot A \cdot X = A^{-1} \cdot D \Rightarrow IX = A^{-1} D \Rightarrow X = A^{-1} \cdot D$

$$A^{-1} = \frac{1}{|A|} \cdot adj(A)$$

Cofactor of $x_{ij} = (-1)^{i+j} \times M_{ij}$

$$5_{11} = (-1)^2 [(26 \times 11 - 4)] = +282$$

$$3_{12} = (-1)^3 [33-14] = -19$$

$$7_{13} = (-1)^4 [6-182] = \cancel{+}176$$

$$3_{21} = -[33-14] = -19$$

$$26_{22} = +[55-49] = 6$$

$$2_{23} = -[10-21] = 11$$

$$7_{31} = +[6-182] = -176$$

$$2_{32} = 2_{23} = 11$$

$$11_{33} = +[-9+130] = +121$$

$$|A| = 5(26 \times 11 - 4) - 3(33-14) + 7(6-182)$$

$$= 121$$

$$\therefore [A]^{-1} = \frac{1}{121} \begin{bmatrix} 282 & -19 & 176 \\ -19 & 6 & 11 \\ -176 & 11 & +121 \end{bmatrix} =$$

$$\therefore X = [A]^{-1} \cdot [D]$$

$$= \frac{1}{121} \begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & +121 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{15.19}\ 0.6336 \\ 0.2727 \\ 0.0000 \end{bmatrix}$$ ✓

* Crammer's Rule:-

Consider a system of eq^n ;

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Then by Crammer's Rule,

$$x=\frac{\Delta x}{\Delta}\ ,\quad y=\frac{\Delta y}{y}\ ;\quad z=\frac{\Delta z}{\Delta}$$

where
$$\Delta x=\begin{vmatrix} d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3\end{vmatrix}$$

$$\Delta y=\begin{vmatrix} a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3\end{vmatrix}$$

$$\Delta z=\begin{vmatrix} a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\end{vmatrix}$$

& $$\Delta=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$$

Pro: Solve by Crammer's Rule,

$$x-3y+z=2$$
$$3x+y+z=6$$
$$5x+y+3z=3$$

$$\therefore \Delta=\begin{vmatrix} 1 & -3 & 1\\ 3 & 1 & 1\\ 5 & 1 & 3\end{vmatrix}=12\qquad \Delta x=\begin{vmatrix} 2 & -3 & 1\\ 6 & 1 & 1\\ 3 & 1 & 3\end{vmatrix}=52\qquad \Delta y=\begin{vmatrix} 1 & 2 & 1\\ 3 & 6 & 1\\ 5 & 3 & 3\end{vmatrix}=-14$$

$$\Delta z=\begin{vmatrix} 1 & -3 & 2\\ 3 & 1 & 6\\ 5 & 1 & 3\end{vmatrix}=-70$$

$$\therefore x=\frac{\Delta x}{\Delta}=\frac{52}{12}=4.33\dot{3}\quad y=\frac{\Delta y}{\Delta}=\frac{-16}{12}=-1.1667\quad z=\frac{\Delta z}{z}=\frac{-70}{12}=-5.8333$$

* Solution of linear algebraic Eqⁿ.

① Gauss Elimination Mtd :- / Upper triangular Matrix / Backward ~~Substitution~~ mtd
② Gauss Jordan Mtd : / Unit Matrix
③ Iteration Mtd :- / Gauss Seidal Mtd.

① Gauss Elimination method:-

In this mtd, the given system of linear eqⁿ is reduced to an upper triangular matrix, which can be solved by back substitution method.

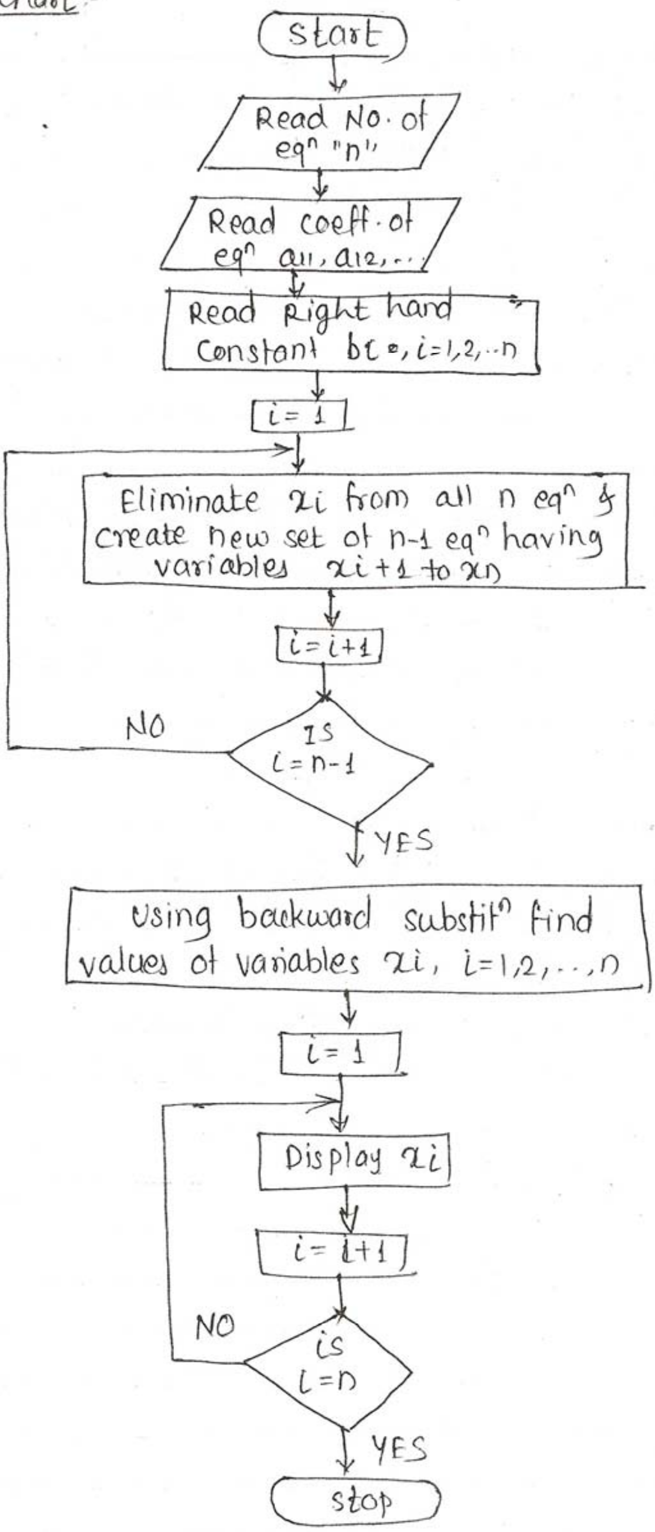
Thus in this method, the matrix is reduced in upper triangular matrix by row or column operations. Thus last unknown is obtained first & then by substituting this value in previous eqⁿˢ, we get value of unknowns.

~~② Gauss Jord~~ Algorithm:

① Read the No. of eqⁿ say 'n'.
② Read the coeff. of matrix A for 'n' eqⁿ (i.e. $a_1, a_2, \ldots$)
③ Read the constant associated with ~~…~~ each eqⁿ say 'b' ($b_1, b_2, \ldots$)
④ Find the augmented matrix (containing value of A & b)
⑤ Go from eliminating the variable from each eqⁿ
⑥ Use backward substitution to find the values of all the variables. x_is
⑦ Display the values of x_is
⑧ Stop.

* Flowchart.

Flowchart:-



Start
Read No. of eqⁿ "n"
Read coeff. of eqⁿ a11, a12, ..
Read Right hand constant bi, i=1,2,..n
i = 1
Eliminate xi from all n eqⁿ & create new set of n-1 eqⁿ having variables xi+1 to xn
i = i+1
NO
IS i = n-1
YES
Using backward substitⁿ find values of variables xi, i=1,2,..,n
i = 1
Display xi
i = i+1
NO
is i=n
YES
stop

② Gauss Jordon Method :-

In this method, back substitution is avoided by reducing the matrix in to unit matrix. So we get unknowns at one step.

③ Gauss seidal Method / Method of iteration :- / Indirect mtd

When a given system of linear eq^n is of large size, Gauss model is not practicable to handle it. Thus to avoid such inpractiability method of $iterat^n$ is used,

In this method approximate values of unknown is assume & it continues to find the closer approximation till we get desired accuracy.

Algorithm :-

Step 1] Read No. of simultaneous eq^n say "n"

Step 2] Read the matrix a [i j] nn the $coeff^t$ of eq^n & b [i]n the right hand constant.

Step 3] Read K = No. of iteration.

Step 4] let x [i]n the value of variables $x_1, x_2, \ldots, x_n$ be all zero's as initial $approximat^n$.

Step 5] Calculate next iteration as,

$$x_1' = \frac{b'}{a_{11}} [b_{11} - (a_{21}x_2 + a_{31}x_3 + \cdots + a_{n1}x_n)]$$

Put this value of x_1' obtained to get next approx. value x_2 as,

$$x_2 = \frac{1}{a_{22}} [b_{22} - (a_{11}x_1' + a_{31}x_3 + \cdots + a_{n1} \cdot x_n)]$$

& so on for all x_i

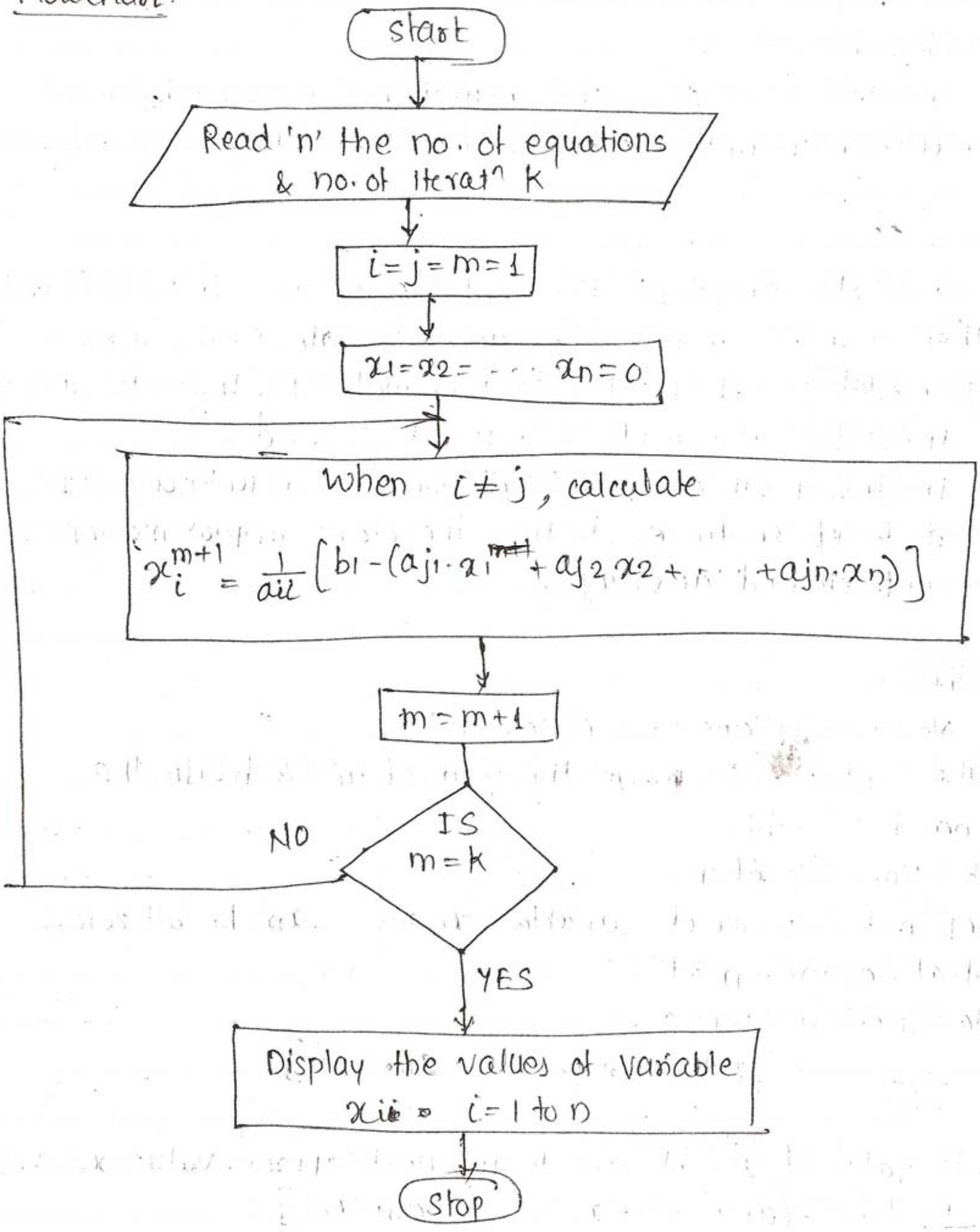
In general $x_i^m = \left[\frac{a}{ii}\right[b_i - (a_{ji}x_1 + a_{j2} \cdot x_2 + \cdots a_{jn}x_n)]$ for $i \neq j$

Step 6] Repeat step 5, K times, the No. $iterat^n$ reqd.

Step 7] Display the values of variable x_i i = 1 to n

Step 8] Stop

Flowchart:-



start
Read 'n' the no. of equations
& no. of Iterat^n K
i=j=m=1
x1=x2= ... xn=0
when i≠j, calculate
$x_i^{m+1} = \frac{1}{a_{ii}}\left[b_1-(a_{j1}\cdot x_1^{m+1}+a_{j2}x_2+\dots+a_{jn}\cdot x_n)\right]$
m=m+1
IS
m=k
NO
YES
Display the values of variable
xi = i=1 to n
Stop