Theory of Structures

Notes by-

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Analysis of Frame | Indeterminate Beam by principle of Least Work or 2^{nd} Theorem of Castigliano

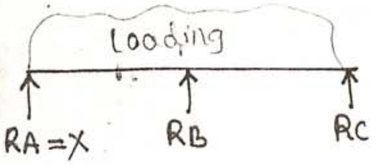
Statically Indeterminate str.	Statically determinate str.
① Eq^m conditions are not sufficient to fully analyse the str.	① $cond^n$ of eq^m are sufficient to fully analyse the str.
② Force response (i.e. BM/SF) } depends on } matl. & c/s of $sect^n$.	Force Response : Independent on properties of matl.
③ Stresses induces due to temp. variation & lack of fit.	③ No stresses induced due to temp. change or lack of fit.

Statically indeterminate str. are analysed by Principle of least work or Second theorem of castigliano.

* Second Theorem of castigliano : ... V.Imp. for MPSC.

"In any & every case of statically indetermination wherein, an indefinite number of different values of the redundant forces satisfies the $cond^n$ of statical eq^m, their actual values are those that render the total strain energy stored to a min."

Explaination:-



Consider a continuous beam ABC, loaded as shown which is statically indeterminate.

as, Degree of statical indetermancy = ~~R-3~~

No. of unknown − No. of eq^n eq^n.

$DSi = 3 - 2$ ⟶ $[\Sigma f_x = 0]$ − inactive.

$= 1.$

∴ To analyse above str., if we assume any value of any unknown reaction (as $DSi = 1$) ; we can evaluate remaining 2.

But we can assume ∞ values of unknown (Say RA) & we get ∞ reactions of remaining unknowns. (i.e. RB & RC)

Our problem is to determine "exact" value amoung all.

This can be done by one additional eq^n i.e. Deflection at $A = 0$

$\therefore y_A = 0.$

But from castigliano's first theorem,

$\boxed{y_A = \frac{\partial W_i}{\partial x}}$ Where. W_i = strain energy stored in the str. which is function of "redundant" quantity X i.e. RA.

Therefore, $cond^n$ $\frac{\partial W_i}{\partial X} = 0$ is the $cond^n$ for max. or min. value of 'X'.

In general, strain energy stored by str. subjected to bending or axial loading is given by,

$$\boxed{W_i = \sum \frac{M^2 ds}{2EI} + \sum \frac{S^2 L}{2AE}}$$

EI = Flexural Rigidity
AE = Axial Rigidity

for detail see "SOM" notes

M = BM at any section
S = axial force in any member.

First Part of above eqⁿ ⟹ strain energy is stored due to bending.
second Part of eqⁿ ⟹ strain energy is stored due to axial loading.

$\therefore$ For, W_i $\begin{smallmatrix}max\\min\end{smallmatrix}$; $\frac{\partial W_i}{\partial X} = 0$

i.e. $\sum \int \frac{ds}{EI} \cdot M \cdot \frac{\partial M}{\partial X} + \sum \frac{L}{AE} \cdot S \cdot \frac{\partial S}{\partial X} = 0$ ... Diff.

i.e. $\frac{\partial^2 W_i}{\partial X^2} = \sum \int \frac{ds}{EI}\left[M \cdot \frac{\partial^2 M}{\partial X^2} + \left(\frac{\partial M}{\partial X}\right)^2\right] + \sum \frac{L}{AE}\left[S \cdot \frac{\partial^2 S}{\partial X^2} + \left(\frac{\partial S}{\partial X}\right)^2\right]$

... Diff. again w.r.t. x.

But $\frac{\partial M}{\partial x}$ & $\frac{\partial S}{\partial x}$ are constant. $\therefore \frac{\partial^2 M}{\partial x^2} = \frac{\partial^2 S}{\partial x^2} = 0$

& $\left(\frac{\partial M}{\partial x}\right)^2$ & $\left(\frac{\partial S}{\partial x}\right)^2$ are positive terms.

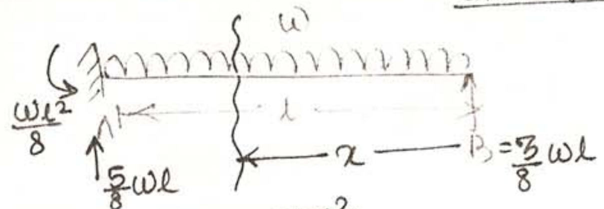
Hence $\boxed{\frac{\partial W_i}{\partial X^2} = \text{Positive}}$ → Law of differentiation (Partial) (See Maths Notes)

$\therefore \frac{\partial W_i}{\partial X} = 0$ is the condⁿ for 'MINIMUM value of W_i

In short
$\therefore$ Amoung ∞ values of X; the value of X which satisfies the eqⁿ condⁿ is the value for which strain energy stored is minimum.

Sample Problem:

GATE Que



$BM_x = R_B x - \frac{wx^2}{2}$

strain energy stored in beam $= W_i = \int \frac{M^2 dx}{2EI}$

$\therefore W_i = \int_0^l \left(R_B \cdot x - \frac{wx^2}{2}\right)^2 \cdot \frac{dx}{2EI}$

By second theorem of Castigliano's R_B shall have such value that strain energy is minimum.

$\therefore$ i.e. $\dfrac{\partial W_i}{\partial R_B} = 0$

$$\therefore \int_0^l 2\left(R_B x - \frac{w x^2}{2}\right)\cdot x \frac{dx}{2EI} = 0 \Rightarrow \int_0^l (2R_B x - w x^2)\cdot x \frac{dx}{2EI} = 0$$

$$\therefore \frac{B}{EI} \Rightarrow \int_0^l 2R_B x \cdot \frac{x\,dx}{2EI} - \int_0^l w x^2 \cdot x \frac{dx}{2EI} = 0$$

$$\Rightarrow \frac{R_B}{EI}\int_0^l x^2 dx - \frac{w}{2EI}\int_0^l x^3 dx = 0$$

$$\Rightarrow \frac{R_B}{EI}\left[\frac{x^3}{3}\right]_0^l - \frac{w}{2EI}\left[\frac{x^4}{4}\right]_0^l = 0$$

$$\Rightarrow \frac{R_B l^3}{3EI} - \frac{w l^4}{8EI} = 0$$

$$\Rightarrow \boxed{R_B = \frac{3wl}{8}}$$

& Mmt. at $A = M_A = \dfrac{wl^2}{8}$

[Three moment Eq^n]

& $R_A + R_B = wl$

$$\therefore R_A = wl - \frac{3}{8}wl = \frac{5}{8}wl \Rightarrow \boxed{R_A = \frac{5}{8}wl}$$

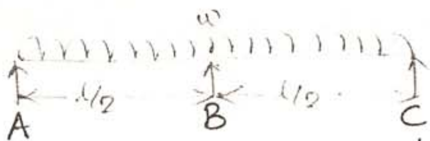
$\Sigma M_B = 0$

$$\therefore lR_B - M_A - \frac{wl^2}{2} = 0$$

$$\therefore M_A = \frac{5}{8}wl^2 - \frac{wl^2}{2}$$

$$\therefore \boxed{M_A = \frac{wl^2}{8}}$$

Pb:



Analyse the beam by principle of least work.

Assume $R_B = R$

$\therefore$ Reaction at A = React^n at C $= \dfrac{wl - R}{2}$

$\therefore BM_x$ from A $= \dfrac{wl - R}{2}\cdot x - \dfrac{wx^2}{2}$ ... Sect^n (AB)

Total strain energy stored $= W_i = \Sigma\int \dfrac{M^2 dx}{2EI}$ ($\times 2$) → Due to 2 spans.

$$\therefore W_i = \Sigma 2 \int_0^{l/2} \frac{1}{2EI}\left[\frac{wl-R}{2}\cdot x - \frac{wx^2}{2}\right]^2 \cdot dx$$

$$\therefore W_i = \frac{1}{EI}\int_0^{l/2}\left[\frac{wl-R}{2}\cdot x - \frac{wx^2}{2}\right]^2 \cdot dx$$

For value of R_B, such that, strain energy is min.

i.e. $\frac{\partial W_i}{\partial R} = 0$

$$\therefore \frac{\partial W_i}{\partial R} = \frac{1}{EI}\int_0^{l/2} 2\left[\frac{wl-R}{2}x - \frac{wx^2}{2}\right]\left[\frac{\partial}{\partial R}\left(\frac{wl\cdot x}{2}\right) - \frac{\partial}{\partial R}\left(\frac{R}{2}\cdot x\right) - \frac{\partial}{\partial R}\left(\frac{wx^2}{2}\right)\right]dx = 0$$

$$= \frac{1}{EI}\int_0^{l/2} 2\left[\frac{wl-R}{2}\cdot x - \frac{wx^2}{2}\right]\left[0 - \frac{x}{2} - 0\right]\cdot dx = 0$$

$$= \frac{1}{EI}\int_0^{l/2} \cancel{2}\left[\frac{wl-R}{2}\cdot x - \frac{wx^2}{2}\right]\left(\frac{-x}{\cancel{2}}\right)dx = 0$$

$$= \frac{1}{EI}\int_0^{l/2}\left[-\frac{wlx^2}{2} + \frac{Rx^2}{2} + \frac{wx^3}{2}\right]dx = 0$$

$$= \frac{1}{EI}\left[-\frac{wl\left(\frac{l}{2}\right)^3}{3\times 2} + \frac{\frac{R}{2}\left(\frac{l}{2}\right)^3}{3} + \frac{\frac{w}{2}\left(\frac{l}{2}\right)^4}{4}\right] = 0$$

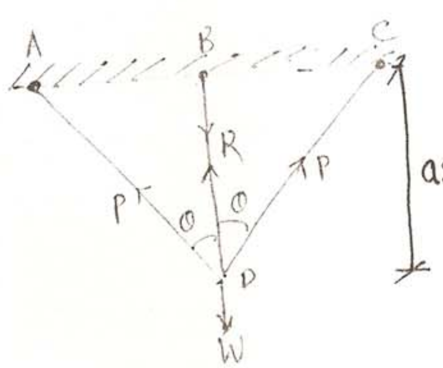
$$= \frac{1}{EI}\left[-\frac{wl^4}{48} + \frac{Rl^3}{48} + \frac{wl^3}{128}\right] = 0$$

$$\therefore \frac{Rl^3}{48} = \frac{wl^4}{48} - \frac{wl^4}{128} = \frac{\cancel{wl^3}}{\cancel{16}} \; \frac{5}{384} wl^4$$

$$\therefore \boxed{R = \frac{5}{8}wL}$$

$$\therefore \boxed{R_A = R_C = \frac{wl - \frac{5}{8}wl}{2} = \frac{3}{16}wl}$$

Axial force



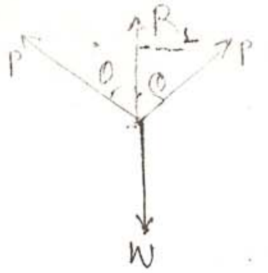
assumed Unity.

$l(AD) = l(CD)$

Area of each wire is same.

Determine the tension in each body.

Soln : Let R be the tension in middle wire BD.
Let P be the tension in wires AD & CD.



$$\therefore \quad 2P\cos\theta = W - R \quad (\Sigma F_y = 0)$$

$$\therefore P = \frac{W-R}{2\cos\theta}.$$

strain energy stored by the wires =

$$W_i = \Sigma \frac{S^2 l}{2AE} \quad [\Sigma S = \Sigma \text{ axial force}]$$

$$\Sigma S^2 l = R^2 l + \frac{P^2}{\cos\theta} + \frac{P^2}{\cos\theta}$$

$$\cancel{\therefore W_i = \Sigma \frac{l}{2AE}\left[S^2\right] = \frac{l}{2AE}\left[(R)^2 + 2(P\cos\theta)^2\right]}$$

$$\Sigma S^2 l = R^2 \cdot l + P^2 \times AD + P^2 \cdot CD$$

But $AD = \frac{l}{\cos\theta}$

$$\text{But } \Sigma S^2 l = R^2 + \frac{2P^2}{\cos\theta}$$

$$\therefore W_i = \frac{l}{2AE}\left[R^2 + \frac{2}{\cos\theta}\left(\frac{W-R}{2\cos\theta}\right)^2\right]$$

$$= \frac{l \cdot R^2}{2AE} + \frac{(W-R)^2}{2\cos^3\theta} \cdot \frac{l}{2AE}$$

$$= \frac{R^2 \cdot l}{2AE} + \frac{(W-R)^2 \cdot l}{4AE\cos^3\theta}$$

R is given by the condition, stored energy (strain) is min.

$$\text{i.e. } \frac{\partial W_i}{\partial R} = 0$$

$$\therefore \frac{\partial W_i}{\partial R} = \frac{l}{2AE}\left(\frac{\partial R^2}{\partial R}\right) + \frac{l}{4AE\cos^3\theta} \cdot \frac{\partial}{\partial R}(W-R)^2 = 0$$

$$\Rightarrow \frac{l}{2AE} 2R + \frac{l}{4AE\cos^3\theta} 2(W-R)(-1) = 0$$

$$\Rightarrow \frac{\cancel{l}R}{\cancel{AE}} - \frac{\cancel{l}(W-R)}{2\cancel{AE}\cos^3\theta} = 0$$

$$\Rightarrow R - \frac{W-R}{2\cos^3\theta} = 0$$

$$\Rightarrow R - \frac{W}{2\cos^3\theta} + \frac{R}{2\cos^3\theta} = 0$$

$$\Rightarrow R\left(1 + \frac{1}{2\cos^3\theta}\right) = \frac{W}{2\cos^3\theta}$$

$$\Rightarrow R = \frac{W/2\cos^3\theta}{\frac{2\cos^3\theta + 1}{2\cos^3\theta}} = \frac{W}{2\cos^3\theta + 1}$$

$$\therefore \boxed{R = \frac{W}{2\cos^3\theta + 1}}$$

Beam ~~Frame~~
10

$$P = \frac{W}{2\cos\theta} - \frac{R}{2\cos\theta} \quad \text{... original eq}^n$$

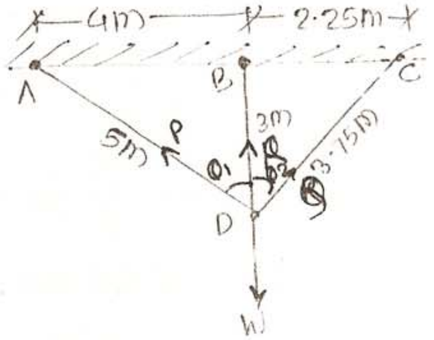
$$= \frac{W}{2\cos\theta} - \frac{W}{2\cos\theta(2\cos^3\theta+1)} = \frac{W}{2\cos\theta}\left[1 - \frac{1}{2\cos^3\theta+1}\right]$$

$$= \frac{W}{2\cos\theta}\left[\frac{2\cos^3\theta+1-1}{2\cos^3\theta+1}\right] = \frac{2W\cos^3\theta}{2\cos\theta(2\cos^3\theta+1)}$$

$$\therefore \boxed{P = \frac{W\cos^2\theta}{2\cos^3\theta+1}}$$

]
2
~~HDOR~~
$D + P^2 CD$

Pro:-



Wires of same matl. & same c/s area.
i.e. AE = constant.
Analyse the str.
(Find Tension in each wire)

NOTE

$D_{si} = 3 - 2 = 1$ → Mtd. used for only, $D_{SC} = 1$
unknown - eq^n eq^n.

Let tension in AD = P, BD = R, CD = Q
i.e. convert all forces in terms of 'R'.
We assume 'R' as "Redundant." (Note: we can assume any force as Redundant)

$\therefore \Sigma F_x = 0 \Rightarrow$

$\angle ADB = \tan^{-1}(4/3) = 53.13 = \theta_1$

$\angle CDB = \tan^{-1}\left(\frac{2.25}{3}\right) = 36.87 = \theta_2$

$\sin\theta_1 = 4/5$; $\sin\theta_2 = \frac{2.25}{3.75}$

$\cos\theta_1 = 3/5$ $\quad \cos\theta_2 = \frac{3}{3.75}$

$\therefore \Sigma F_x = 0 \quad \therefore P\sin\theta_1 - Q\sin\theta_2 = 0$

$$\therefore P = \frac{Q\sin\theta_2}{\sin\theta_1} = Q\left(\frac{2.25/3.75}{4/5}\right)$$

$$\therefore \boxed{P = \frac{3}{4}Q}$$

$\Sigma F_y = 0 \Rightarrow P\cos\theta_1 + R + Q\cos\theta_2 = W$

$$\therefore \frac{3}{4}Q\left(\frac{3}{5}\right) + R + Q\left(\frac{3}{3.75}\right) = W$$

$$\therefore \frac{9}{20}Q + R + \frac{3}{3.75}Q = W \Rightarrow \frac{4Q}{5} + R = W$$

$\therefore \cancel{Q = W - \frac{4}{5}R} \Rightarrow \boxed{Q = \frac{4}{5}(W-R)}$

$\therefore P = \frac{3}{4}Q = \frac{3}{4}\times\frac{4}{5}(W-R)$

$\therefore \boxed{P = \frac{3}{5}(W-R)}$

$\therefore$ Strain energy stored $= W_i = \sum \frac{S^2 l}{2AE}$ (AE = constant)

$\therefore W_i = \frac{1}{2}\sum \frac{S^2 l}{\cancel{2}}$ But $\sum S^2 l = P^2(AD) + R^2(BD) + Q^2(CD)$

$\therefore \sum S^2 l = P^2\times 5 + R^2\times 3 + Q^2\times 3.75$

$\cancel{\therefore W_i = \frac{1}{2}\left[\left(\right.\right.}$

$$\therefore W_i = \frac{1}{2}\left\{5\left[\frac{3}{5}(W-R)\right]^2 + 3R^2 + 3.75\left[\frac{4}{5}(W-R)\right]^2\right\}$$

For exact value of R; $\frac{\partial W_i}{\partial R} = 0$.

$$\therefore \frac{\partial W_i}{\partial R} = \frac{1}{2}\cdot\frac{\partial}{\partial R}\left\{5\times\left(\frac{3}{5}\right)^2\frac{\partial}{\partial R}(W-R)^2 + 3\frac{\partial}{\partial R}(R^2) + 3.75\left(\frac{4}{5}\right)^2\cdot\frac{\partial}{\partial R}(W-R)^2\right\} = 0$$

$$\Rightarrow \frac{1}{2}\times\frac{9}{5}\times 2(W-R)(-1) + \frac{3}{2}(2R) + \frac{12}{2\times 5}\times 2(W-R)(-1) = 0$$

$$\Rightarrow -\frac{9}{5}(W-R) + 3R - \frac{24}{10}(W-R) = 0$$

$$\Rightarrow -\frac{9}{5}W - \frac{24}{10}W + \frac{9}{5}R + 3R + \frac{24}{10}R = 0$$

$$\Rightarrow R = \frac{(21/5)W}{(36/5)} = \frac{(21/5)W}{(36/5)}$$

$\therefore \boxed{R = \frac{7W}{12}}$ .... good ..... Ans.

$\therefore Q = \frac{4}{5}\left(W - \frac{7W}{12}\right) \Rightarrow \boxed{Q = \frac{1}{3}W}$ ... Ans.

$\therefore P = \frac{3}{5}\left(W - \frac{7}{12}W\right) \Rightarrow \boxed{P = \frac{1}{4}W}$ ... Ans.

Extra Note

Let, $\delta l_P = \frac{P.l}{AE} = \frac{1}{4}\frac{W\times 5}{AE} = \frac{5}{4}\frac{W}{AE}$

$\delta l_Q = \frac{F.l}{AE} = \frac{1}{3}\frac{W\times 3.75}{AE} = \frac{3.75}{3}\frac{W}{AE}$

$\delta l_R = \frac{F.L}{AE} = \frac{7}{12}\times\frac{W\times 3}{AE} = \frac{7}{4}\frac{W}{AE}$

Elongation or extension of each wire.

$\boxed{dL = \frac{P.L}{AE}}$

See "SOM" Notes.