Theory of Structures

Notes by-

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Moment Distribution Method

(E)

Mtd. evolved by : Hardy Cross

Used for : Analysis of statically indeterminate beams & frames with Rigid jts.

Carry over Theorem :-

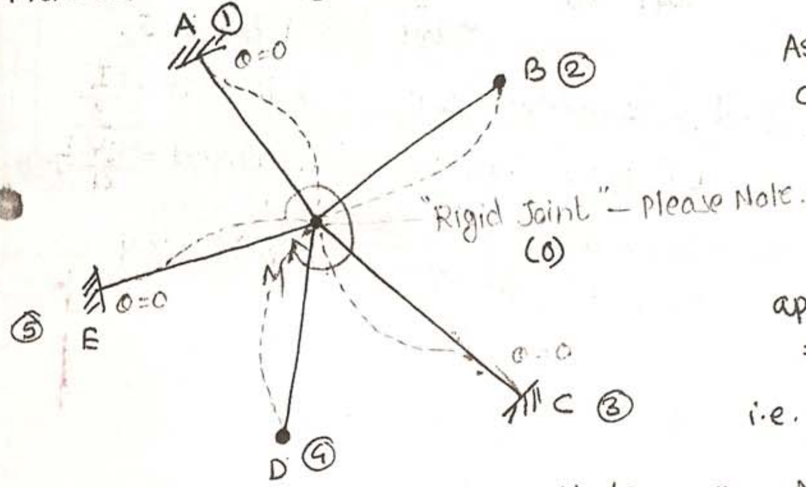
When a moment 'M' is applied to produce a rotation without translation at the near supported end 'B' of a beam whose farther end 'A' is fixed, the carry over moment 'Ma' at the farther end is $\frac{1}{2}$th the applied mmt. 'M' & is of same sense (order) as that of applied moment.

Stiffness of member :- When a structural member of uniform section is subjected to a moment at one end only, then the moment reqd. so as to rotate that end to produce unit slope, is called as stiffness of the member.

For proped cantilever,	For simply supported beam,
$\text{stiffness} = \frac{4EI}{l}$	$\text{stiffness} = \frac{3EI}{l}$

i.e. $M = \frac{4EI\,\theta_b}{l}$ $\qquad M = \frac{3EI\,\theta_B}{l}$ $\qquad (\theta_B = \text{Unit})$

Distribution Theorem :- A moment which is applied to a structural joint to produce rotation without translation, gets distributed amoung the connecting members at the joints in the same proportion as their stiffness.



As members are "rigidly" connected at 'O', the slope due to applied moment will be same for each member.

i.e. $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5$ ∵ at O

∵ With usual notions,

applied moment = M

= Balancing moments.

i.e. $M = M_1 + M_2 + M_3 + M_4 + M_5$

∴ But $\theta_1 = \frac{M_1 l_1}{4E_1I_1}$ (Fixed end), $\theta_2 = \frac{M_2 l_2}{3E_2I_2}$ (Hinged end) $\quad \theta_3 = \frac{M_3 l_3}{4E_3I_3} \quad \theta_4 = \frac{M_4 l_4}{3E_3I_4}, \theta_5 = \frac{M_5 l_5}{4E_5I_5}$

But $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5$ ∵ O is Rigid joint.

$$\therefore \frac{M_1 l_1}{4E_1I_1} = \frac{M_2 l_2}{3E_2I_2} = \frac{M_3 l_3}{4E_3I_3} = \frac{M_4 l_4}{3E_4I_4} = \frac{M_5 l_5}{4E_5I_5}$$

$$\therefore \frac{M_1}{\left[\frac{4E_1I_1}{l_1}\right]} = \frac{M_2}{\left[\frac{3E_2I_2}{l_2}\right]} = \frac{M_3}{\left[\frac{4E_3I_3}{l_3}\right]} = \frac{M_4}{\left[\frac{3E_4I_4}{l_4}\right]} = \frac{M_5}{\left[\frac{4E_5I_5}{l_5}\right]}$$

$\therefore M_1 : M_2 : M_3 : M_4 : M_5 = \frac{4E_1I_1}{l_1} : \frac{3E_2I_2}{l_2} : \frac{4E_3I_3}{l_3} : \frac{3E_4I_4}{l_4} : \frac{4E_5I_5}{l_5}$

i.e. Moment is distributed in the same proportion of their stiffness.

$\therefore$ If $\sigma_1 = \frac{4E_1I_1}{l_1}$; $\sigma_2 = \frac{3E_2I_2}{l_2}$, $\sigma_3 = \frac{4E_3I_3}{l_3}$, $\sigma_4 = \frac{3E_4I_4}{l_4}$; $\sigma_5 = \frac{4E_5I_5}{l_5}$

$\therefore \frac{M_1}{\sigma_1} = \frac{M_2}{\sigma_2} = \frac{M_3}{\sigma_3} = \frac{M_4}{\sigma_4} = \frac{M_5}{\sigma_5} = \frac{M}{\sigma}$ → $\left.\frac{\text{Applied mmt}}{\text{Total stiffness}}\right\}$ at jt. O.

$\therefore M_1 = \left(\frac{\sigma_1}{\sigma}\right) M$, $M_2 = \left(\frac{\sigma_2}{\sigma}\right) M$, $M_3 = \left(\frac{\sigma_3}{\sigma}\right) M$, $M_4 = \left(\frac{\sigma_4}{\sigma}\right) M$, $M_5 = \left(\frac{\sigma_5}{\sigma}\right) M$.

Where $\boxed{\left(\frac{\sigma_1}{\sigma}\right) = \text{Distribution factor for the member OA at jt. O}}$

& so on...

* Distribution factor :- The distribution factor for a member at a joint is the ratio of stiffness of the member to the total stiffness of all the members meeting at that joint.

* Relative stiffness :-

Let ; $M_1 : M_2 : M_3 : M_4 : M_5 = \frac{4E_1I_1}{l_1} : \frac{3E_2I_2}{l_2} : \frac{4E_3I_3}{l_3} : \frac{3E_4I_4}{l_4} : \frac{4E_5I_5}{l_5}$

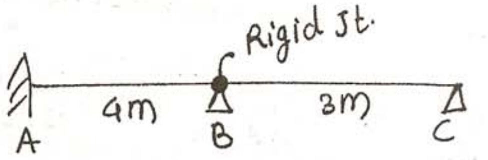
$= \frac{I_1}{l_1} : \frac{3}{4}\frac{I_2}{l_2} : \frac{I_3}{l_3} : \frac{3}{4}\frac{I_4}{l_4} : \frac{I_5}{l_5}$

The ratios, $\frac{I_1}{l_1}, \frac{3}{4}, \frac{I_2}{l_2}, \frac{I_3}{l_3}, \frac{3}{4}, \frac{I_4}{l_4}, \frac{I_5}{l_5}$ ⇒ Relative Stiffness.
of member 1, 2, 3, 4, 5.

$\therefore$ Relative stiffness of a member at a jt. whose farther end is fixed = $\frac{I}{l}$
Relative stiffness of a member at a jt. whose farther end is hinged = $\frac{3}{4}\frac{I}{l}$

$\boxed{\therefore \text{D.F.} = \frac{\text{Relative stiffness of member}}{\text{Total Relative stiffness at a jt.}}}$ ⇐ Convinient way

for eg:-



$I_{ab} : I_{bc} = 1 : 2$

$\therefore$ Stiffness of BA $= \sigma_{ba} = \frac{4EI_{ab}}{l_{ab}} = \frac{4EI}{4} = EI$

Stiffness of BC $= \sigma_{bc} = \frac{3EI_{bc}}{l_{bc}} = \frac{3E(2I)}{3} = 2EI$

Total stiffness at joint B $= \sigma = \sigma_{ba} + \sigma_{bc} = EI + 2EI = 3EI$

$\therefore$ D.F. for BC at B $= \frac{\sigma_{ba}}{\sigma} = \frac{EI}{3EI} = \frac{1}{3}$

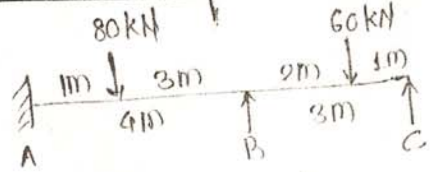
D.F. for BA at B $= \frac{\sigma_{bc}}{\sigma} = \frac{2EI}{3EI} = \frac{2}{3}$

MD 2

ie. More systematically,

Joint	Member	Relative stiffness (I/L)	Total Relative stiffness	DF
B	BA	(1) $\frac{I_{ab}}{L} = \frac{I}{4}$ Farther end fixed	$\frac{3I}{4}$	$\frac{I/4}{3I/4} = \frac{1}{3}$
	BC	$\left(\frac{3}{4}\right)\frac{I_{bc}}{l} = \frac{3}{4} \times \frac{2I}{3} = \frac{2I}{4}$ Farther end hinged.		$\frac{I/2}{3I/4} = \frac{2}{3}$

Pro: 1



$I_{ab} : I_{bc} = 2:1$

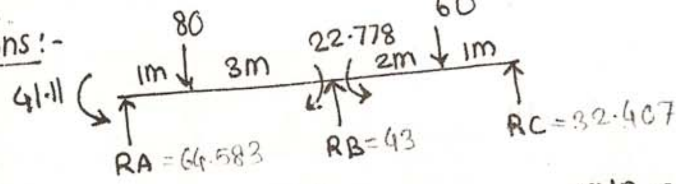
Soln:- Step I) Distribution Factor :-

Joint	member	Relative stiffness	Total Relative stiffness	D.F.
B	BA	$\frac{I}{L} = \frac{2I}{4}$	$\frac{3}{4}I$	2/3
	BC	$\frac{3I}{4L} = \frac{3(2I)}{4\times3}$		1/3

Step II) Final End Moments :

Joint	A	B		C
Member	AB	BA	BC	CB
DF		2/3	1/3	
FEM	−45	+15	−40/3	+80/3
Bal			−40/3	−80/3
Modified ~~Final~~ FE mmt	−45	+15	−80/3	0
Bal		+7.788	+3.889	
CO	3.889			
Final mmt	−41.111	+22.778	−22.778	0

Step III) Reactions :-



$\therefore \Sigma M_B = 0$ ... (LHS)

$\therefore 4R_A - 41.11 + 22.778 - 80 \times 3 = 0$

$\Rightarrow R_A = 64.583$

$\Sigma M_B = 0$ ... (RHS)

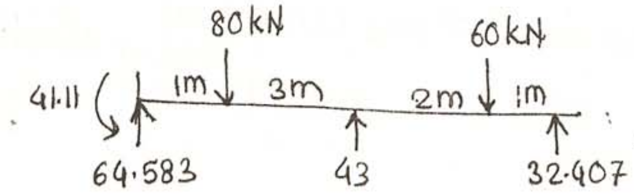
$\therefore 3R_C = 60 \times 2 - 22.778$

$\therefore R_C = 32.407$

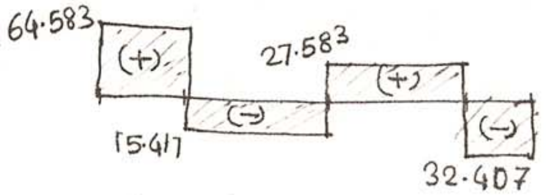
$\Sigma F_y = 0 \Rightarrow R_B = 80 + 60 - 64.583 - 32.407$

$= 43$ kN.

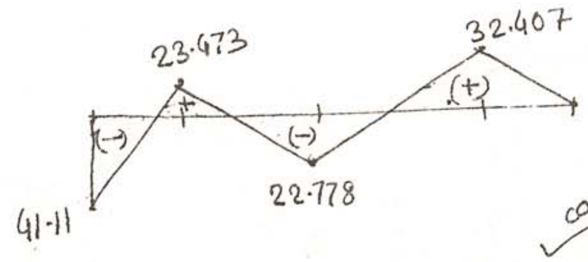
step V] SFD & BMD



SFD (kN)

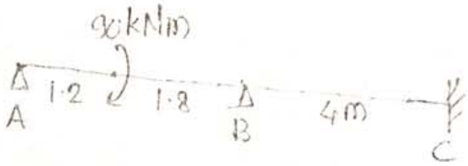


BMD (kNm)



correct.

Pro: 2



$\overline{M}ab = \frac{M \cdot b(3a-l)}{l^2} = 10.80\,kNm$

$\overline{M}ba = \frac{M \cdot a(3b-l)}{l^2} = 28.80\,kN$

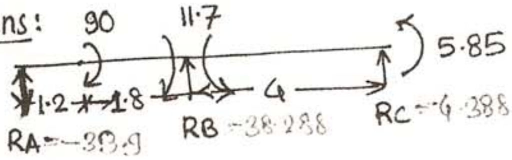
step I] Distribution Factor:-

Joint	Member	Relative stiffness	Total Relative stiffness	D.F.
B	~~A~~BA	$\frac{3}{4}\frac{I}{l} = \frac{3}{4}\frac{I}{3} = \frac{I}{4}$	0.5I	0.5
	BC	$\frac{I}{l} = \frac{I}{4}$		0.5

step II] Final end moments:-

Joint	A	B		C
Member	AB	BA	BC	
DF	—	0.5	0.5	—
FEM	+10.80	28.80	0	0
Bal	−10.80	−5.40		
Modified FEM	0	+25.4	0	
Bal	0	−11.7	−11.7	0
CO				−5.850
Final mmts	0	+11.7	−11.7	−5.850

Step III) Reactions:



$\Sigma M@B=0$ (LHS)

$\therefore 3RA + 90 + 11.7 = 0$

$RA = -33.9$

$\Sigma M@B=0$ (RHS)

$\therefore -5.85 - 11.7 - Rc \times 4 = 0$

$\therefore Rc = 4.388$

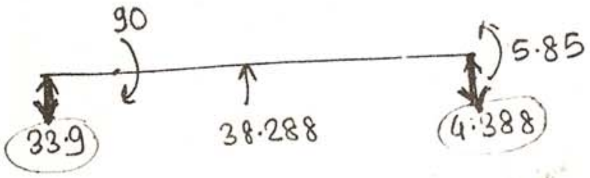
$\Sigma M@C=0$

$\therefore -33.9 \times 7 + 90 + 4RB - 5.85 = 0$

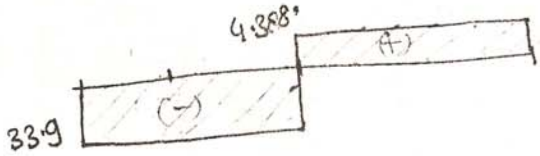
$\therefore RB = 38.288$ kN

Step IV) SFD & BMD :-

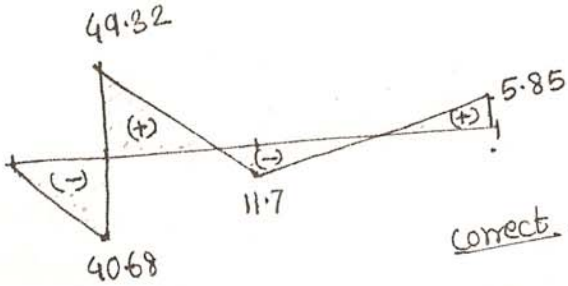
Loading Diagram (Analysed)



SFD (kN)

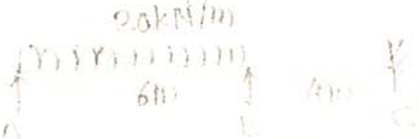


BMD (kNm)



Correct.

Pro: 2).



step I) Distribution factor:-

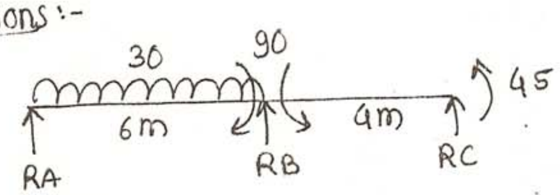
Joint	Member	Relative stiffness	Total relative stiffness	D.F.
B	BA	$\frac{3}{4}\frac{I}{L} = \frac{3}{4}\times\frac{I}{6}$	$\frac{3I}{8}$	$1/3$
	BC	$\frac{I}{L} = \frac{I}{4}$		$2/3$

step II) Final end moments:-

Joint	A	B		C
Member	AB	BA	BC	CB
DF	—	1/3	2/3	—
FEM	−90	+90	0	0
Bal	+90	+45	0	0
Modified FEM	0	+135	0	0
Bal		−45	−90	−45
Bal		+45	+90	+45
Bal		−45	−90	−45
Final EM	0	+90	−90	−45

Extra Balancing. Please Note

step III) Reactions:-



$\Sigma M @ B = 0$ (LHS)

$\therefore 6 RA = 30 \times \frac{6^2}{2} - 90$

$\therefore RA = \cancel{105}\ 75$ kN

$RB_1 = $ ~~$RB = 30\times6 - 75 - 33.75$~~
~~$= 71.25$ kN~~

$\Sigma M @ B = 0$ (RHS) $\quad 4RC + 45 + 90 = 0$

$\therefore$ ~~$4RC = +45+90$~~

$\therefore RC = 33.75$ kN.

$\Sigma M @ C = 0$

$\therefore 10 \times \cancel{105}\ 75 - 30\times6\times(3+4) + 4RB - 45 = 0$

$\therefore RB = \cancel{138.75}$ kN $\cdot 63.75$ kN

138.75